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Negin Golrezai and Sourav Sahoo



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Bidding in Uniform Price Auctions for Value Maximizing Buyers

Negin Golrezaei* and Sourav Sahoo†

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Abstract

We study the problem of bidding in uniform price auctions, a widely used format in Treasury auctions, emissions permit auctions, and energy markets. Although uniform price auctions are non-truthful for bidders with quasilinear utility functions, several empirical findings suggest that this auction format induces truthful bidding from the bidders. We attribute this difference in theory and practice to the assumption of the behavioral model of the bidders. In this pursuit, we study uniform price auctions in a repeated setting from the perspective of a single *value-maximizing* buyer who aims to maximize their acquired cumulative value across T rounds, subject to per-round return-on-investment (RoI) constraints. For a RoI-constrained and value-maximizing buyer, we study a generalized version of the uniform bidding format, commonly used in practice, which we term as *m-uniform bidding*. Under *m-uniform bidding*, the buyer submits m pairs of bid and quantity values (b_i, q_i) , demanding q_i units at bid b_i . To characterize the optimal *m-uniform* bid, we introduce and study the notion of universally feasible (UF) bidding policies, which are robust, meaning that RoI feasibility is obtained regardless of the competitors' bids. We show that the optimal class of UF bidding policies is essentially a *generalization of truthful bidding policies*, which depends only on the valuation curve of the bidder and target RoI, irrespective of the bids submitted by competitors. To measure the performance of UF bidding policies against the optimal bidding policy that is not necessarily UF, we introduce a metric called the Price of Universal Feasibility (PoUF) and establish that PoUF is at most 2, irrespective of m , and show that the upper bound is tight. We further compare the generalized *m-uniform* bidding interface against the classical uniform bidding format under which $m = 1$, showing the total value under *m-uniform* bidding increases at most by a factor of m . Numerical simulations on semi-synthetic data demonstrate that UF bidding policies perform significantly better than the derived theoretical bounds, and this, combined with their straightforward characterization, makes them highly appealing in practice.

1 Introduction

We study uniform price auctions, a widely adopted multi-unit auction format in various domains, including Treasury auctions, emissions permit auctions, and energy markets. In this auction, the auctioneer offers K identical units of a commodity to a group of buyers (bidders), each of whom may demand multiple units. The auctioneer collects bids from the participants, allocates the K units to the K highest bids, and determines the per-unit auction clearing price, denoted as p , by setting it equal to the K^{th} highest bid.

The practical appeal of this auction format often stems from a few key aspects: (i) its equitable pricing, as the per-unit price remains the same among all successful bids (which is also necessary in

*Sloan School of Management, Massachusetts Institute of Technology. Email: golrezae@mit.edu

†Operations Research Center, Massachusetts Institute of Technology. Email: sourav99@mit.edu

certain cases since antitrust laws prohibit price discrimination in wholesale markets (Cramton and Stoft, 2006)), (ii) reduction in market manipulation (Chari and Weber, 1992), and (iii) the perception of simplified (truthful) bidding strategies (Friedman, 1959; Kahn et al., 2001; Klemperer, 2009). While the first two assertions hold true, the last claim does not align with the classical quasilinear utility behavior of bidders under which uniform price auctions are known to be non-truthful. Hence, bidders are susceptible to demand reduction (Milgrom, 2004) and are more likely to bid strategically. However, several empirical studies on bidding behaviors in uniform price auctions indicate that the format induces truthful bidding (Cason and Plott, 1996; Hailu and Thoyer, 2010) and suppresses demand reduction in the presence of a large number of bidders (Engelbrecht-Wiggans et al., 2006). These empirical findings suggest that the conventional model of bidders’ behavior is not entirely applicable in this context.

Traditional mechanism design has primarily focused on the agents whose utilities are *quasilinear* in their payments, i.e., an agent’s utility linearly decreases in its payments. However, in several practical scenarios, agents deviate significantly from such behavior. Such deviations may arise because agents can have different disutility of the money spent, budget, or return-on-investment (RoI) constraints. A particular category of non-quasilinear agents is *value maximizers*. As the name suggests, these agents aim to maximize the value obtained in auctions subject to performance constraints (such as RoI), resource constraints (such as budgets), or a combination of both. See Section 1.3 for more details.

There are several contexts in which it is prudent to consider agents as value maximizers instead of classical utility maximizers. For example, in microeconomic consumer choice theory, individuals aim to maximize the value function (in consumer theory, the term *utility* function is used instead of value function) subject to budget constraints (Mas-Colell et al., 1995, pp. 50). The concept of value maximization has also gained considerable attention in recent times, particularly in the realm of online advertising markets (for related citations, see Deng et al. (2020); Lucier et al. (2023); Balseiro et al. (2021a); Deng et al. (2023b)). Here, advertisers strive to maximize their total value, such as the number of clicks, while ensuring the average cost per click remains below a specified threshold.

We aim to leverage the notion of value maximizers to understand bidders’ behavior in uniform price auctions. Formally, we posit that bidders in a uniform price auction aim to acquire the most value while adhering to certain (soft) financial constraints. As we will explain in more detail later, under this model, optimal bidding strategies for bidders possess a straightforward structure that can be regarded as an extension of truthful bidding, thereby confirming the prevailing perceptions of this auction format.

Bidding for Value Maximizing Buyers. In this work, as one of our modeling contributions, we consider a repeated setting where in each round $t \in [T]$, a uniform price auction with K units is conducted. In this context, we study the bidding problem from the perspective of a single bidder who aims to maximize their cumulative value over T rounds, subject to per-round RoI constraints, in which the ratio of per-unit acquired value to per-unit price should exceed a certain constant, referred to as the “target RoI”.

Practical Bidding Interface: m -Uniform Bidding. With the objective of addressing practical bidding interfaces, where each bidder submits a limited number of bid-quantity pairs rather than a separate bid for each unit demanded (as illustrated in Fig. 1 for the New Zealand Emissions Trading Scheme (NZ ETS)), we study a generalized version of the uniform bidding format (De Keijzer et al., 2013; Birmipas et al., 2019) which we term as m -uniform bidding. In m -uniform bidding, denoted as $\mathbf{b} := \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$ with $b_1 > b_2 > \dots > b_m$, bidders bid b_1 for the first q_1 units, b_2 for the next q_2 units, and so on.

Optimal Bidding Policies: Generalizing Truthful Bidding. Under this practical bidding format, we aim to characterize an optimal class of (value-maximizing) RoI-feasible m -uniform bidding

Figure 1: Bidding Interface for submitting one bid-quantity pair in NZ ETS Auctions. To submit m bid-quantity pairs, the bidder needs to repeat the process m times. See more details at <https://www.etsauctions.govt.nz/public/training/participate>.

policies. Within this class, we require the policies to be universally feasible (UF); see Definition 2. Under a UF policy, RoI feasibility is maintained even under arbitrary competing bids, providing robustness to the bidding strategy. In sealed bid auctions, where individual bid data is kept private, such a property is essential for formulating feasible bids with limited information about competitors. Furthermore, it is also necessary for designing online learning algorithms, ensuring that bidders can submit RoI-feasible bids without knowledge of competitors' bids. See Section 7 for more details.

1.1 Our Contributions

Optimal Universally Feasible Class of Bidding Policies. In Section 4, as our first result, we show that the optimal class of UF bidding policies, denoted by \mathcal{P}_m^* , can be viewed as a natural extension of truthful bidding, depending only on the bidder's valuation vector $[v_1, v_2, \dots, v_K]$ with diminishing returns (i.e., $v_1 \geq v_2 \geq \dots \geq v_K$) and target RoI, γ :

Theorem 1.1 (Informal: Generalized Truthful Bidding). *Let $\mathbf{b} = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$ be a m -uniform bidding policy, and define $Q_\ell = \sum_{j=1}^\ell q_j$, $\ell \in [m]$, as the maximum number of demanded units in the first ℓ bid-quantity pairs. The optimal UF class of all k -uniform bidding policies, where $k \in [m]$, denoted by \mathcal{P}_m^* , is given by:*

$$\mathcal{P}_m^* = \left\{ \mathbf{b} = \langle (b_1, q_1), \dots, (b_k, q_k) \rangle : b_\ell = \frac{w_{Q_\ell}}{1 + \gamma}, \forall \ell \in [k], \forall k \in [m] \right\}.$$

Here $w_\ell = \frac{1}{\ell} \sum_{j=1}^\ell v_j$ is the average per-unit value of the first ℓ units.

The theorem demonstrates that once the bidder determines the quantities q_1, \dots, q_m , the best action is to submit $w_{Q_1}/(1 + \gamma)$ as the first bid, which represents the normalized average per-unit value of the first $Q_1 = q_1$ units. Subsequently, for the second bid, one should submit $w_{Q_2}/(1 + \gamma)$, which is the normalized average per-unit value of the first $Q_2 = q_1 + q_2$ units, and so forth. This theorem, which further emphasizes the nested structure within the optimal UF bidding policies as illustrated in Figure 2, reinforces the conventional wisdom that bidding in uniform price auctions should follow a truthful pattern. Specifically, the optimal UF class of bidding policies depends solely on the valuation vector $[v_1, \dots, v_K]$ and γ .

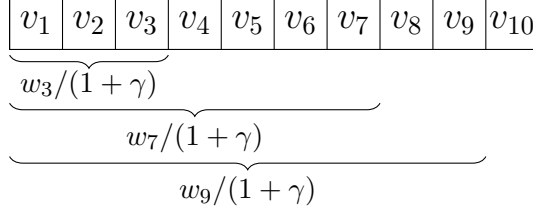


Figure 2: In the figure, we highlight the nested structure of the bidding policies in \mathcal{P}_m^* . Suppose that $K = 10$ and $m = 3$. Consider the nested bidding policy $\mathbf{b} = \left\langle \left(\frac{w_3}{1+\gamma}, 3 \right), \left(\frac{w_7}{1+\gamma}, 4 \right), \left(\frac{w_9}{1+\gamma}, 2 \right) \right\rangle \in \mathcal{P}_m^*$, where we note that $Q_1 = 3$, $Q_2 = 3 + 4 = 7$ and $Q_3 = 3 + 4 + 2 = 9$. The j^{th} highest bid (i.e., b_j) is the average of the first Q_j coordinates of the value vector (scaled by $(1 + \gamma)$).

Price of Universal Feasibility (RoI-Robustness). To characterize the optimal bidding policies, we enforce the desirable property of the bidding policies being UF. This leads to robust bidding policies that are RoI feasible regardless of the competing bids. However, one might be curious about the cost incurred when enforcing the bidding policies to be UF. In Section 5, we quantify this cost by introducing a concept called the price of universal feasibility, which is the maximum ratio of the value obtained by an optimal bidding policy that is not necessarily UF to the value of the optimal UF bidding policy. We show that:

Theorem 1.2 (Informal: Price of Universal Feasibility). *The price of universal feasibility is upper bounded by 2, and this bound is tight.*

Generalized m -Uniform Bidding versus Classical Uniform Bidding (with $m = 1$). In Section 6, we study the impact of generalizing the classical uniform bidding format, where the bidder submits a single pair of bid and quantity. It is evident that by increasing the number of bid-quantity pairs, as done in m -uniform bidding, one anticipates an increase in the bidder’s obtained value. In this work, we characterize this gain and show that, in the best case, the value can exhibit linear growth with respect to m . The main result of this section is:

Theorem 1.3 (Informal: m -Uniform Bidding versus Classical Uniform Bidding). *For any $m > 1$, the ratio of the optimal value under the optimal m -uniform bidding to that under the classical uniform bidding (with $m = 1$) is upper-bounded by m , and this bound is tight.*

Discussion on Learning to Bid. In Section 7, we discuss how the bidder can identify the optimal (value-maximizing) bidding policy within the class \mathcal{P}_m^* . For a constant m , off-the-shelf no-regret online learning algorithms can help identify the optimal policy efficiently. However, such methods can become computationally intractable for a large value of m . To this end, we also propose an efficient (polynomial time) method for identifying an $(1 - 1/e)$ -approximate solution by leveraging the structure of the bidding policies.

Numerical Simulations. In Section 8, we conduct numerical simulations on semi-synthetic data generated from aggregate statistics of the past EU ETS auctions. We observe that the UF bidding policies perform significantly better in practice than the theoretical bounds presented in Section 5 and Section 6. These experiments suggest that bidders can obtain near-optimal performance (in terms of obtained value) while adhering to simple UF bidding strategies in uniform price auctions.

1.2 Key Technical Challenges

The core challenge in our work stems from the interdependence of the values obtained by the constituent bid-quantity pairs in a m -uniform bidding policy. Recall that a m -uniform bidding

policy consists of m bid-quantity pairs of the form (b_i, q_i) . In the considered setting, the values obtained by the individual bid-quantity pairs in the bidding policy are interdependent; i.e., even for a fixed competing bid profile, the value obtained by any bid-quantity pair (b_j, q_j) depends on the remaining $m - 1$ bid-quantity pairs in the policy. As a key technical contribution (see Lemma 3 and a stronger version in Lemma 8), we show that the value obtained by a m -uniform RoI feasible bid (see definition in Section 2.1) can be expressed as the maximum of the value obtained by m independent 1-uniform feasible bids (b'_j, q'_j) . This result decouples the dependency equipping us to approximate the value obtained by the optimal m -uniform UF policy using 1-uniform UF policies as components; which is crucial for establishing the upper bounds in Section 5 and Section 6. In Section 7, we utilize it to learn an efficient approximate optimal solution in the online setting.

The second key challenge lies in obtaining matching lower bounds for Theorem 1.2 and Theorem 1.3 (formally stated as Theorem 5.2 and Theorem 6.1 respectively). Formally, to demonstrate that the ratio obtained for a problem instance is within an interval of a (small) δ from the upper bound, we need to consider a problem instance with $T = \Theta(1/\delta^m)$ auctions, each selling $K = \Theta(1/\delta^m)$ units. As the number of auctions and the number of bidding policies become exponentially large in m , the construction of the problem instance and its analysis are quite non-trivial. The construction of the problem instance crucially leverages the result describing the value obtained by 1-uniform bids when compared with the optimal bidding policy (see Lemma 9 for the result and, Appendix B.2 and Appendix B.3 for the construction of the problem instances).

1.3 Related Work

Value Maximizers and RoI Constraints. The concept of agents as value maximizers within financial constraints is a well-established notion in microeconomic theory (Mas-Colell et al., 1995, pp. 50). In mechanism design literature, one of the earliest explorations of value-maximizing agents was conducted by Wilkens et al. (2016). Their work primarily delved into the single-parameter setting, characterizing truthful auctions for value maximizers. Similarly, Fadaei and Bichler (2016) and Lu et al. (2023) studied truthful (approximate) revenue-maximizing mechanisms in combinatorial markets tailored for such agents.

In recent years, there has been a surge of interest in RoI-constrained value maximizers, particularly in the realms of autobidding and online advertising (Aggarwal et al., 2019; Balseiro et al., 2021a; Deng et al., 2021; Balseiro et al., 2021b, 2022; Deng et al., 2023a,b; Golrezaei et al., 2023; Liaw et al., 2023). See Golrezaei et al. (2021c) as one of the earliest studies on auction design for RoI-constrained buyers, validating such soft financial constraints using data from online advertising auctions.

Our contribution to this body of research lies in the study of value-maximizing buyers in uniform price auctions under RoI constraints. We demonstrate that these buyers can effectively employ nested bidding strategies that depend only on their valuation vector.

Combinatorial Auctions. Our research contributes to the extensive body of literature concerning combinatorial auctions, as evidenced by works by De Vries and Vohra (2003); Pekeć and Rothkopf (2003); Blumrosen and Nisan (2007); Palacios-Huerta et al. (2022). In this work, we focus on auctions for identical goods, which find widespread application in various practical scenarios, including Treasury auctions (Garbade and Ingber, 2005; Binmore and Swierzbinski, 2000; Nyborg et al., 2002), procurement auctions (Cramton and Ausubel, 2006), wholesale electricity markets (Tierney et al., 2008; Fabra et al., 2006), and emissions permit auctions (Goulder and Schein, 2013; Schmalensee and Stavins, 2017; Goldner et al., 2020). Several works have focused on designing truthful multi-unit auctions (in a utility-maximizing sense) that achieve a good approximation of social welfare and/or revenue (Dobzinski and Nisan, 2015; Dobzinski and Leme, 2014; Borgs et al., 2005). Our work contributes to this literature by studying uniform price auctions from the

perspective of bidding algorithms for value-maximizing buyers.

Bidding in Auctions and Auction Parameter Optimization. Traditionally, the problems of bidding in auctions and refining auction parameters have been studied under Bayesian settings (Myerson, 1981; Hartline and Roughgarden, 2009; Beyhaghi et al., 2021; Riley and Samuelson, 1981). However, any bidder rarely has an exact characterization of the private valuations (or other parameters) of its competitors (Wilson, 1985). Consequently, in recent years, there has been a growing interest in studying these problems under data-driven settings, for both offline and online contexts. Examples include Sandholm (2003) for automated mechanism design using valuation samples, Roughgarden and Wang (2019); Derakhshan et al. (2022, 2021); Golrezaei et al. (2021a) for using offline or online data to set reserve prices in VCG auctions, Balseiro et al. (2019) for online problems in bidding in first-price auctions, and Golrezaei et al. (2021b) for optimizing boost values of boosted second-price auctions using historical auction bids.

In this line of work, the closest to our work is by Brânzei et al. (2023a); Galgana and Golrezaei (2023). They designed the optimal bidding algorithms for agents with quasilinear utilities in multi-unit uniform price and pay-as-bid auctions on the offline data and leveraged these offline algorithms to obtain their online counterparts with no-regret property. In our work, as one of our focus, we also design an optimal class of bidding policies in multi-unit uniform price auctions. However, as the main difference, we focus on value-maximizing RoI-constrained buyers.

2 Model

We consider uniform price multi-unit auctions for identical items. There are n buyers (bidders) indexed by $i \in [n]$, and K identical items. Each bidder i has a fixed private value for each of the items, denoted by $\mathbf{v}_i \in \mathbb{R}_{\geq 0}^K$. The valuations are assumed to be diminishing across the items, i.e., $v_{i,1} \geq v_{i,2} \geq \dots \geq v_{i,K}$. The maximum total demand, denoted by $\overline{M} \in [K]$, is then defined as $\min\{j \in [K-1] : v_{j+1} = 0\}$. If such an index does not exist, we set \overline{M} as K . For each bidder i with valuation vector $\mathbf{v}_i = [v_{i,1}, v_{i,2}, \dots, v_{i,K}]$, we define an *average cumulative valuation* vector as follows

$$\mathbf{w}_i = [w_{i,1}, w_{i,2}, \dots, w_{i,K}], \quad \text{where} \quad w_{i,j} = \frac{1}{j} \sum_{\ell \leq j} v_{i,\ell}, \forall j \in [K]. \quad (1)$$

As $v_{i,1} \geq v_{i,2} \geq \dots \geq v_{i,K}$, we also have $w_{i,1} \geq w_{i,2} \geq \dots \geq w_{i,K}$.

2.1 Uniform Price Auctions and Bidders' Behaviour

Auction Format. Each bidder i submits a bid vector $\mathbf{b} \in \mathbb{R}_{\geq 0}^{\overline{M}}$. The bids submitted by other bidders are denoted as β_{-i} , and we define $\beta := (\mathbf{b}, \beta_{-i})$.

- *Allocation Rule.* The auctioneer collects the bids from all the bidders, sorts the bids in non-increasing order, and allocates items to the bidders with the top K bids. That is, if bidder i has j bids in the top K positions, they are allocated j items. We assume that the bidders submit bids from a continuous space; hence, there are no ties almost surely.
- *Payment Rule.* The auctioneer follows the last-accepted-bid (LAB) payment rule (bidders pay the K^{th} highest bid per unit). An alternative pricing rule is the first-rejected-bid (FRB) payment rule (agents pay the $(K+1)^{\text{th}}$ highest bid per unit). The per-unit price is also referred to as the clearing price. In this work, we primarily focus on the LAB payment rule as it is more common in practice and has more appealing properties (see Brânzei et al. (2023a)), but the results can be extended for the FRB payment rule with careful modifications.

For a bid profile let β , $x_i(\beta)$ and $p(\beta)$ denote the number of units allocated to bidder i and the clearing price respectively. So, the total value obtained, $V_i(\beta)$, and total payments paid, $P_i(\beta)$, is:

$$V_i(\beta) = \sum_{j \leq x_i(\beta)} v_{i,j} \quad \text{and} \quad P_i(\beta) = p(\beta) \cdot x_i(\beta). \quad (2)$$

Repeated Setting. In practice, several multi-unit auctions, such as emission permit auctions, happen in a repeated fashion. Repeated auctions allow bidders to learn from the actions of all participants in previous rounds and improve their bidding policy successively. In this work, we primarily focus on this repeated setting. Formally, the auction described in the previous paragraph takes place sequentially in T rounds indexed by $t \in [T]$. Let β_{-i}^t denote the bids submitted by all bidders except bidder i in round t and $\beta_{-i,t}^{-(j)}$ be the j^{th} smallest winning bid in the absence of bids from bidder i for round t . The bid history $\mathcal{B}_{-i} = [\beta_{-i}^1, \dots, \beta_{-i}^T]$ contains bids submitted by all bidders except bidder i for the past T auctions. In round t , suppose the bidder i submits a bid \mathbf{b}^t and the bid profile is $\beta^t := (\mathbf{b}^t; \beta_{-i}^t)$. Let $x_i(\beta^t)$ and $p(\beta^t)$ be the number of units allocated to bidder i and the clearing price, respectively, in round t . Hence, the total value obtained is $V_i(\beta^t) = \sum_{j \leq x_i(\beta^t)} v_{i,j}$, and total payment is $P_i(\beta^t) = p(\beta^t) \cdot x_i(\beta^t)$.

Value Maximizing Bidders. The objective of the bidders is to maximize their individual total value while adhering to a constraint that ensures the total value obtained is at least a constant multiple of the payments made to acquire those items. This constraint can be equivalently expressed as a return-on-investment (RoI) constraint. We assume that the target RoI (i.e., the multiplier), denoted as $\gamma \geq 0$, is both private and fixed for each bidder i . Formally, to derive an optimal bidding policy, a value-maximizing bidder solves Problem (VM).

$$V^*(\mathcal{B}_{-i}) = \max_{\mathbf{b} \in \mathbb{R}_{\geq 0}^M} \sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t) \quad (\text{VM})$$

such that $V_i(\mathbf{b}; \beta_{-i}^t) \geq (1 + \gamma)P_i(\mathbf{b}; \beta_{-i}^t), \quad \forall t \in [T].$

Here, $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$. If in round t , the RoI constraint is violated, we define $V_i(\mathbf{b}; \beta_{-i}^t) = -\infty$. We say a bid vector is *feasible* if the RoI constraints hold true for each round.

In problem (VM), we enforce the RoI constraints for each auction individually rather than as an aggregate constraint over T rounds, as is typically assumed in online ad auctions (Deng et al., 2021, 2022; Feng et al., 2023). Lucier et al. (2023) consider constraints similar to us, i.e., the value obtained is at least a constant times the payment in each auction for autobidders which they term as *marginal RoI (or value) constraint*. In this work, we drop the prefix ‘marginal’ for the sake of brevity. Trivially, if the constraints for (VM) hold true for each round, they also hold true in aggregate. The rationale behind imposing constraints for each round is that, unlike ad auctions that often occur simultaneously and frequently, multi-unit auctions in the context of Treasury and emission permit auctions typically take place sequentially over much longer time horizons. It is therefore intuitive that bidders aim to maintain profitability in each auction, rather than waiting for a potentially indefinite period to cover their losses¹.

We conclude this section by noting the resemblance between the offline optimization problem in (VM) and those studied in Roughgarden and Wang (2019), Derakhshan et al. (2022), Galgana and Golrezaei (2023), and Brânzei et al. (2023a). Roughgarden and Wang (2019) focus on optimizing

¹Although EU ETS emission permit auctions are scheduled to occur regularly, regulations stipulate that an auction may be canceled if the bidders’ demand falls short of the supply of permits or if the auction clearing price does not meet the reserve prices (Regulations, 2019). Hence, the bidders are more likely to ensure feasibility in each round.

reserve prices in VCG auctions with access to historical bid prices, while Brânzei et al. (2023a) and Galgana and Golrezaei (2023) derive optimal bidding policies for quasilinear bidders in multi-unit auctions with similar information available. Both studies demonstrate that efficiently solving the problem and gaining insights into its structure in the offline setting is instrumental in designing no-regret online learning algorithms. For a more in-depth exploration of leveraging offline algorithms to create no-regret learning algorithms, we refer readers to Roughgarden and Wang (2019); Niazadeh et al. (2022); Brânzei et al. (2023a).

2.2 Bidding in Uniform Price Auctions

Most of the research literature on multi-unit auctions assumes that bidders follow the standard bidding format, wherein they submit a vector of individual bids, with each bid corresponding to one unit (Brânzei et al., 2023a; Babaioff et al., 2023; Birmpas et al., 2019; Galgana and Golrezaei, 2023). In contrast, several works (e.g., (De Keijzer et al., 2013; Birmpas et al., 2019)) study uniform bidding, bidders submit bids in the form of (b, q) pairs, where a bidder bids b for the first q items and zero for the remaining units. For a bid-quantity pair, (b, q) , we equivalently state that the bidder has a demand q at bid b . We study a natural generalization of the uniform bidding policy, which we term as m -uniform bidding, for any integer $m \geq 1$.

Definition 1 (m -Uniform Bidding). *For a fixed $m \in \mathbb{Z}_{>0}$,*

- *a m -uniform bid is characterized by m bid-quantity pairs, denoted as $\mathbf{b} := \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$, where, without loss of generality, we assume that $b_1 > b_2 > \dots > b_m > 0$ and $q_j > 0$, $j \in [m]$.*
- *We introduce the notation $\mathbf{b}[1 : \ell] = \langle (b_1, q_1), \dots, (b_\ell, q_\ell) \rangle$, for all $\ell < m$, to represent the first ℓ bid-quantity pairs within a m -uniform bid $\mathbf{b} = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$.*
- *We further define $Q_j = \sum_{\ell=1}^j q_\ell$ for all $j \in [m]$ as the total quantity demanded in the first j bid-quantity pairs, with $Q_0 = 0$, where we assume, without loss of generality, that $Q_m \leq \overline{M}$.²*

Note that any set of m bid-quantity pairs, such as $\mathbf{b} = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$, can be equivalently expressed as a vector. Therefore, we use a notation similar to that employed for bid vectors to represent sets of bid-quantity pairs.

The m -uniform bidding format can be viewed as a special case of the bidding language for product-mix auctions when only a single variety of goods is present (Klemperer, 2009). This format is also described as the piece-wise linear bidding language in Nisan (2015). If $m = \overline{M}$, the bidding format is equivalent to standard bidding. However, in practice, bidders often submit only a few bid-quantity pairs. For instance, in the EU ETS emission permit auctions for 2023, bidders submitted an average of approximately 4.35 bid-quantity pairs per auction (EEX, 2023).

In fact, our generalization is motivated by practical settings, such as the aforementioned EU ETS emission auctions and the Treasury auctions, where the number of individual units K can be quite large. Submitting a bid vector listing bid values for each individual unit becomes prohibitive in such cases. Similarly, using a single bid, as done in uniform bidding (which corresponds to a 1-uniform bidding policy), to represent the entire valuation curve can be misleading and restrictive, particularly if the curve exhibits sharp decay across the items. Therefore, the m -uniform bidding policies offer a flexible and computationally efficient approach for submitting bids, a method widely utilized in practice (see an example in Fig. 1).

We illustrate how the auction works in the considered setting in Example 1.

²Suppose the bidder bids for, and wins more than \overline{M} units. There is no additional value being allocated over \overline{M} units, but the total payment increases (assuming the clearing price is positive), potentially violating the RoI constraint.

Example 1. Consider a single auction. There are $n = 2$ bidders and $K = 5$ identical items. The valuations are: $\mathbf{v}_1 = [6, 4, 3, 1, 1]$ and $\mathbf{v}_2 = [5, 3, 1, 1, 0]$. Both the bidders are value maximizers with target RoI, $\gamma_1 = \gamma_2 = 0$. Suppose $m = 2$ and the bids submitted by the bidders are $\mathbf{b}_1 = \langle (5, 2), (3, 3) \rangle$ and $\mathbf{b}_2 = \langle (4, 2), (2, 2) \rangle$. The bids in sorted order are: $[5, 5, 4, 4, 3, 3, 3, 2, 2]$. So, bidder 1 is allocated 3 items, and bidder 2 gets 2 items. The market clearing price is 3. Hence, $V_1(\beta) = 6 + 4 + 3 = 13$, $V_2(\beta) = 5 + 3 = 8$, $P_1(\beta) = 3 \cdot 3 = 9$, and $P_2(\beta) = 2 \cdot 3 = 6$. Note that the RoI constraints for both the bidders are satisfied.

2.3 Universally Feasible Bidding Policies

Our objective in this work is to design an optimal bidding strategy, maximizing value from the perspective of a single bidder denoted as i . This bidding strategy should universally adhere to RoI feasibility, as defined below:

Definition 2 (Universally Feasible). A m -uniform bidding policy, $\mathbf{b} = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$, is considered universally feasible (UF) if it remains feasible for all possible bid histories, $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$. In other words, for any \mathcal{B}_{-i} , we have

$$V_i(\mathbf{b}; \beta_{-i}^t) \geq (1 + \gamma)P_i(\mathbf{b}; \beta_{-i}^t), \quad \forall t \in [T].$$

The collection of ALL UF k -uniform bidding policies, where $k \in [m]$, is denoted as \mathcal{P}_m . Note that the set \mathcal{P}_m is downward closed in the sense that for any $k \leq m$, $\mathcal{P}_k \subseteq \mathcal{P}_m$.

Alternatively put, UF bidding policies ensure that the total value obtained in any single auction is at least a constant (i.e., $(1 + \gamma) \geq 1$) times the total payments for the auction. From the perspective of the bidders, this property is mild, natural and necessary. This property is particularly appealing because under UF bidding strategies, irrespective of the bids submitted by competitors, the bidder consistently maintains RoI feasibility. There is also an inherent necessity for a UF policy class because most sealed bid auctions do not disclose the individual bids *ex post* to protect sensitive financial information and prevent bidding malpractices. Hence, bidders need to formulate feasible bidding policies under limited information about the bids submitted by others. In addition, UF bidding policies are necessary to design online learning algorithms for bidding, where the bidder needs to submit a RoI feasible bid without having access to the competitors' bids (see details in Section 7). This property is essential because a causal, optimal, but non-UF bidding policy may be infeasible in future rounds as highlighted in Example 2.

Example 2. Let $K = 4$, $\gamma = 0$, $\bar{M} = 3$, $m = 1$, and $\mathbf{v} = [9, 5, 1]$, yielding an average cumulative valuation vector $\mathbf{w} = [9, 7, 5]$. This leads to possible UF bidding policies of $(9, 1)$, $(7, 2)$, and $(5, 3)$. Consider an online setting, i.e., the bidder has no information about any future competing bids. If the competing bids in the first two rounds are $\beta_{-i}^1 = [4, 4, 4, 4]$ and $\beta_{-i}^2 = [7, 6.5, 5.5, 5.5]$, the total values obtained by the UF bidding policies are 18, 28, and 15 respectively. Now, consider a non-UF bidding policy: $(6, 3)$, which is RoI feasible (for $t = 1$ and 2) and yields a value of 29 in the first two auctions, surpassing all UF policies. However, if the bidder bids $(6, 3)$ for $t = 3$ when $\beta_{-i}^3 = [7, 5.5, 5.5, 5.5]$, the total value obtained is 15 but the total payment is 18, violating RoI feasibility. This example also illustrates that a non-UF policy can not be learnt in a natural (online) setting.

Having established the concept of UF bidding policies, we now focus on developing optimal UF bidding policies for any $m \in \mathbb{Z}_{>0}$. These policies solve the following problem:

$$V_m^*(\mathcal{B}_{-i}) = \max \sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t) \tag{VM-UNI}_m$$

$$\text{such that } \mathbf{b} = \langle (b_1, q_1), \dots, (b_k, q_k) \rangle \in \mathcal{P}_m \quad \text{and} \quad k \in [m]. \quad (3)$$

Here, with a slight abuse of notation, we denote $V_m^*(\mathcal{B}_{-i})$ as the maximum achievable value when considering UF bidding policies in \mathcal{P}_m , under the bid history, \mathcal{B}_{-i} . Observe that the optimal solution to problem (VM-UNI $_m$) is RoI feasible as $\mathbf{b} \in \mathcal{P}_m$. Let $\mathbf{b}_m^*(\mathcal{B}_{-i})$ be a UF bidding policy in \mathcal{P}_m that achieves the maximum value. We are now equipped to define an *optimal* UF class.

Definition 3 (Optimal Universally Feasible Class). *For any $m \in \mathbb{Z}_{>0}$, we define an optimal UF class and denote it by $\mathcal{P}_m^* \subseteq \mathcal{P}_m$. The optimal class \mathcal{P}_m^* has the following properties:*

Property 1. *For any \mathcal{B}_{-i} of arbitrary size, we have $\mathbf{b}_m^*(\mathcal{B}_{-i}) \in \mathcal{P}_m^*$. If there are multiple $\mathbf{b}_m^*(\mathcal{B}_{-i})$, we require that at least one of such bidding policies is in \mathcal{P}_m^* .*

Property 2. *The optimal class \mathcal{P}_m^* is minimal in the downward closed manner, that is for any k -uniform bid $\mathbf{b} \in \mathcal{P}_m^*$ where $k \in [m]$, there exists a bid history, $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$, for which*

- \mathbf{b} is optimal, i.e., $\sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t) \geq \sum_{t=1}^T V_i(\mathbf{b}'; \beta_{-i}^t), \forall \mathbf{b}' \in \mathcal{P}_m^*$;
- the value obtained under \mathbf{b} and \mathcal{B}_{-i} is strictly larger than that under ANY k' -uniform bid $\mathbf{b}' \in \mathcal{P}_m^*$ and \mathcal{B}_{-i} where $k' \leq k$. That is, $\sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t) > \sum_{t=1}^T V_i(\mathbf{b}'; \beta_{-i}^t), \forall \mathbf{b}' \in \mathcal{P}_m^* \setminus \{\mathbf{b}\}$.

In other words, Property 2 states that the k -uniform bid \mathbf{b} is the unique optimal bidding policy for the bid history \mathcal{B}_{-i} in the restricted class $\mathcal{P}_k^* \subseteq \mathcal{P}_m^*$. While Property 1 ensures that the class \mathcal{P}_m^* contains the optimal UF bidding policy for any bid history, Property 2 implies that \mathcal{P}_m^* is the minimal class satisfying this property, i.e., removing policies from \mathcal{P}_m^* may violate Property 1.

2.4 The Price of Universal Feasibility

While being UF is essential, such a requirement can come at a cost. To quantify this cost, denoted as the ‘Price of Universal Feasibility’ or ‘PoUF’ for short, we consider the following strong benchmark. Under the benchmark, similar to Problem (VM-UNI $_m$), we aim to find a k -uniform bidding policy \mathbf{b} , where $k \in [m]$, that maximizes the total value for a given bid history \mathcal{B}_{-i} . However, unlike Problem (VM-UNI $_m$), we do not require \mathbf{b} to be UF; that is, we do not enforce $\mathbf{b} \in \mathcal{P}_m$. Instead, we only require \mathbf{b} to be RoI feasible under the bid history \mathcal{B}_{-i} .

Formally, our benchmark, denoted by $V_m^{\text{OPT}}(\mathcal{B}_{-i})$, is given as

$$V_m^{\text{OPT}}(\mathcal{B}_{-i}) = \max_{\mathbf{b}} \sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t) \quad (\text{VM-OPT}_m)$$

$$\text{such that } V_i(\mathbf{b}; \beta_{-i}^t) \geq (1 + \gamma)P_i(\mathbf{b}; \beta_{-i}^t), \quad \forall t \in [T]$$

$$\text{and } \mathbf{b} = \langle (b_1, q_1), \dots, (b_k, q_k) \rangle \quad \text{for some } k \in [m].$$

Let the optimal bidding policy be denoted as $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$. We emphasize that $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ is not necessarily feasible for any other bid history except \mathcal{B}_{-i} .

The Price of Universal Feasibility is then defined as

$$\text{PoUF}_m := \max_{\mathcal{B}_{-i}} \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_m^*(\mathcal{B}_{-i})}. \quad (\text{PoUF})$$

This metric is analogous to established metrics in mechanism design, such as the price of anarchy (Koutsoupias and Papadimitriou, 1999), the price of stability (Anshelevich et al., 2008) to measure the inefficiency of equilibrium, and the price of fairness (Bertsimas et al., 2011) to measure the inefficiency due to fair division in resource allocation. Trivially, $1 \leq \text{PoUF}_m < \infty$ ³.

³Suppose for some \mathcal{B}_{-i} , $V_m^{\text{OPT}}(\mathcal{B}_{-i}) = 0$. By definition, $V_m^*(\mathcal{B}_{-i}) = 0$. In this case, we define $\frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_m^*(\mathcal{B}_{-i})} = \frac{0}{0} = 1$.

2.5 Increasing the Number of Bid Quantity Pairs, m

In Section 2.2, we introduced m -uniform bidding and justified its preference over the (1-)uniform bidding format. However, increasing the number of bid quantity pairs m can increase the complexity of bidding (see our discussion in Section 7). Considering this, we aim to characterize the maximum benefit of increasing the number of bid quantity pairs m , compared to the base case of $m = 1$.

To do so, we introduce the following two metrics

$$\text{RATIO-UNI}_m := \max_{\mathcal{B}_{-i}} \frac{V_m^*(\mathcal{B}_{-i})}{V_1^*(\mathcal{B}_{-i})}, \quad \text{RATIO-OPT}_m := \max_{\mathcal{B}_{-i}} \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_1^{\text{OPT}}(\mathcal{B}_{-i})}. \quad (4)$$

The first metric measures, in the best case, how much more value bidder i obtains by submitting an optimal UF k -uniform bid where $k \in [m]$, compared with that an optimal UF 1-uniform bid. The second metric does a similar comparison without restricting the bids to be UF. By definition, $1 \leq \text{RATIO-UNI}_m, \text{RATIO-OPT}_m < \infty$.

3 Characterizing Universally Feasible Policy Classes

In this section, for any m , we characterize the universally feasible (UF) policy class \mathcal{P}_m per Definition 2. This characterization will then be used to establish the optimal UF policy class $\mathcal{P}_m^* \subseteq \mathcal{P}_m$.

We start by generalizing the idea of underbidding (overbidding) in the m -uniform bidding setting for multi-unit auctions. Recall that in single-item auctions, if the private value of the item by a bidder is v , then a bid b is said to be an underbid (overbid) if $b < v$ ($b > v$).

Definition 4 (m -uniform non-truthful bids). *Let \mathbf{w} and γ denote the average cumulative valuation vector and the target RoI respectively. A m -uniform bid $\mathbf{b} = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$*

- is an underbid if

$$b_j \leq \frac{w_{Q_j}}{1 + \gamma}, \quad \forall j \in [m] \quad \text{and} \quad \exists \ell \in [m] \quad \text{such that} \quad b_\ell < \frac{w_{Q_\ell}}{1 + \gamma}.$$

- is an overbid if

$$\exists \ell \in [m] \quad \text{such that} \quad b_\ell > \frac{w_{Q_\ell}}{1 + \gamma}.$$

In other words, assuming that $\gamma = 0$, we define $\mathbf{b} = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$ as an underbid if, for all $j \in [m]$, the bid b_j is less than or equal to the average of the first Q_j coordinates of the value vector, denoted as w_{Q_j} , and there exists $\ell \in [m]$ where the inequality is strict. Here, $Q_j = \sum_{\ell \leq j} q_\ell$ represents the maximum number of demanded units in the first j bids. Recall that for the value vector $[v_1, v_2, \dots, v_K]$, $w_j = \frac{1}{j} \sum_{\ell \leq j} v_\ell$ represents the average of the first j coordinates of \mathbf{v} . Similarly, the bid is an overbid if there exists some $\ell \in [m]$ such that b_ℓ is strictly greater than the average of the first Q_ℓ coordinates of the value vector. Having defined the overbidding and underbidding notions, we are ready to present the main result of this section:

Theorem 3.1. *For any m , no overbidding is allowed in \mathcal{P}_m . That is, the collection of all UF k -uniform bidding policies, where $k \in [m]$, per Definition 2 is given by*

$$\mathcal{P}_m = \left\{ \mathbf{b} = \langle (b_1, q_1), \dots, (b_k, q_k) \rangle : b_\ell \leq \frac{w_{Q_\ell}}{1 + \gamma}, \forall \ell \in [k], \forall k \in [m] \right\}.$$

The detailed proof is deferred to Appendix A.1. As a high-level idea, we show that overbidding is not a UF bidding policy; hence, $\mathcal{P}_m \subseteq \mathcal{P}_m^{\text{NOB}}$, the set of no-overbidding (NOB) policies. We complete the proof by stating that every NOB policy is also UF.

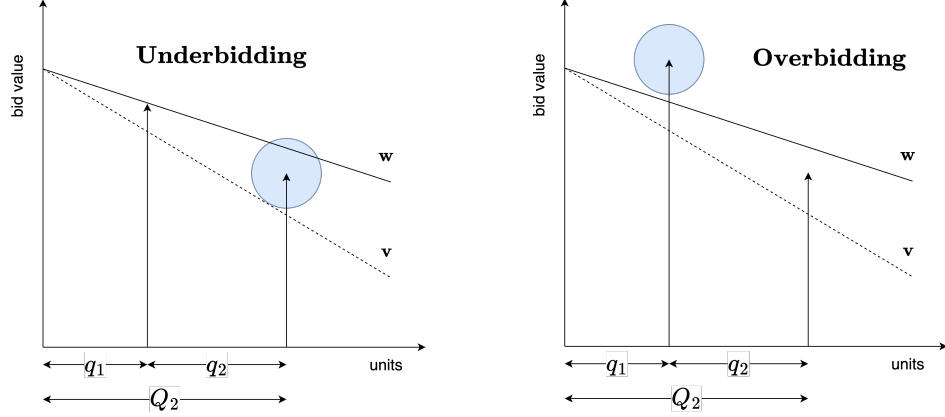


Figure 3: The solid line represents the average cumulative valuation curve and the dotted line represents the valuation curve. The figure in the left (resp. right) illustrates underbidding (resp. overbidding) for a 2-uniform bidding policy. The blue circles highlight the bids that are non-truthful in this setting. Note that the notions of underbidding and overbidding in Definition 4 are defined with respect to the average cumulative valuation curve and *not* the valuation curve. The valuation (and average cumulative valuation) curve are linear in the figure for illustrative purposes only.

4 Characterizing Optimal Universally Feasible Policy Classes

Now that we have established the class of UF bidding policies, \mathcal{P}_m , we proceed to design the *optimal* subclass as described in Definition 3. The following theorem is the main result of this section. This theorem shows that, in contrast to the UF class \mathcal{P}_m , strategies involving underbidding are suboptimal and hence are eliminated from the set \mathcal{P}_m^* .

Theorem 4.1 (The Optimal Universally Feasible Class). *Let $\bar{\mathcal{P}}_m \subseteq \mathcal{P}_m$ be the class of bidding policies obtained by removing the underbidding policies,*

$$\bar{\mathcal{P}}_m = \left\{ \mathbf{b} = \langle (b_1, q_1), \dots, (b_k, q_k) \rangle : b_\ell = \frac{w_{Q_\ell}}{1 + \gamma}, \forall \ell \in [k], \forall k \in [m] \right\}.$$

Then, the optimal class of UF k -uniform bidding policies, where $k \in [m]$, $\mathcal{P}_m^ = \bar{\mathcal{P}}_m$.*

Theorem 4.1 shows that the policies in the optimal UF class have a nested structure illustrated in Fig. 2 in the sense that the j^{th} highest bid (i.e., b_j) is the average of the first Q_j coordinates of the value vector (scaled by $(1 + \gamma)$). We refer to the bid in the form of $\mathbf{b} = \left\langle \left(\frac{w_{Q_1}}{1 + \gamma}, q_1 \right), \dots, \left(\frac{w_{Q_m}}{1 + \gamma}, q_m \right) \right\rangle$ as a “nested” m -uniform bidding policy.

Another implication of Theorem 4.1 is that fixing Q_j ’s uniquely determines the bidding policy. By definition, for policies in the class \mathcal{P}_m^* , fixing Q_j exactly determines $b_j = w_{Q_j}/(1 + \gamma)$. Consequently, the k -uniform bidding policy, where $k \in [m]$, becomes uniquely identified. As we will elaborate in later sections (refer to Section 7 for detailed explanations), this fact plays a crucial role in learning the optimal UF bidding policy within \mathcal{P}_m^* on the fly as more data becomes available over time.

4.1 Proof of Theorem 4.1: The Optimal Universally Feasible Class

Recall from Definition 3 that the optimal UF class has two essential properties:

Property 1. It contains the optimal UF bidding policy for any \mathcal{B}_{-i} , and

Property 2. It is minimal in a downward closed sense to satisfy Property 1 (see Definition 3).

Showing Property 1. We begin by establishing a general result regarding the monotonicity of feasible bid vectors (not necessarily m -uniform bids) for value maximizing agents. As the result holds for any bid vector, it is also true for m -uniform bids.

Lemma 1 (Monotonicity of feasible bids). *Consider two sorted bid vectors: $\mathbf{b} = [b_1, b_2, \dots, b_k]$ and $\mathbf{b}' = [b'_1, b'_2, \dots, b'_k]$ such that $b_j \geq b'_j, \forall j \in [k]$. Suppose \mathbf{b} is RoI feasible for some fixed $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$. Then, for the given \mathcal{B}_{-i} , the value obtained by \mathbf{b} is more than that by \mathbf{b}' , that is, $\sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t) \geq \sum_{t=1}^T V_i(\mathbf{b}'; \beta_{-i}^t)$.*

The proof of Lemma 1 is presented in Appendix A.2. We now leverage Lemma 1 to show Property 1. To do so, we will argue that underbidding is a weakly dominated bidding strategy, i.e., for every underbid \mathbf{b} (per Definition 4), there exists a non-underbidding (NUB) policy $\mathbf{b}' \in \mathcal{P}_m$ such that for all bid histories, $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$, $\sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t) \leq \sum_{t=1}^T V_i(\mathbf{b}'; \beta_{-i}^t)$. To show this, suppose $\mathbf{b} = \langle (b_1, q_1), \dots, (b_k, q_k) \rangle$, where $k \in [m]$, is an underbid. Consider $\mathbf{b}' = \langle (b'_1, q'_1), \dots, (b'_k, q'_k) \rangle$ such that $q'_j = q_j$ and $b'_j = \frac{w_{Q_j}}{1+\gamma}, \forall j \in [k]$. By Theorem 3.1, we establish that $\mathbf{b}, \mathbf{b}' \in \mathcal{P}_m$. Invoking Lemma 1 completes the proof which shows that underbidding is a dominated strategy, and hence such policies can be removed from \mathcal{P}_m to obtain the class of nested m -uniform policies, $\overline{\mathcal{P}}_m$.

Showing Property 2. We will show that $\overline{\mathcal{P}}_m$ satisfies Property 2 implying $\overline{\mathcal{P}}_m = \mathcal{P}_m^*$. We claim that $\overline{\mathcal{P}}_m$ satisfies the conditions (also described in Definition 3) and thus complete the proof.

For any k -uniform bid $\mathbf{b} \in \mathcal{P}_m^*$, where $k \in [m]$, there *exists* a bid history, $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$, for which

- \mathbf{b} is optimal, i.e., $\sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t) \geq \sum_{t=1}^T V_i(\mathbf{b}'; \beta_{-i}^t), \forall \mathbf{b}' \in \mathcal{P}_m^*$,
- the value obtained under \mathbf{b} and \mathcal{B}_{-i} is strictly larger than that under ANY k' -uniform bid $\mathbf{b}' \in \mathcal{P}_m^*$ and \mathcal{B}_{-i} where $k' \leq k$. That is, $\sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t) > \sum_{t=1}^T V_i(\mathbf{b}'; \beta_{-i}^t), \forall \mathbf{b}' \in \mathcal{P}_m^* \setminus \{\mathbf{b}\}$.

Our construction. To show the claim, fix any $\mathbf{b} \in \overline{\mathcal{P}}_m$. Let $\mathbf{b} = \langle (\frac{w_{Q_1}}{1+\gamma}, q_1), \dots, (\frac{w_{Q_k}}{1+\gamma}, q_k) \rangle$. We construct the bid history, $\mathcal{B}_{-i}(\mathbf{b})$ consisting of $T = k$ rounds. In round $t \in [k]$, for a sufficiently small $\epsilon > 0$ and $C \gg w_1$,

$$\beta_{-i,t}^{-(j)}(\mathbf{b}) = \begin{cases} \frac{w_{Q_{t+1}}}{1+\gamma} + \epsilon, & \text{if } 1 \leq j \leq Q_t \\ C, & \text{if } Q_t < j \leq K \end{cases}, \quad \forall t \in [k], Q_t < \overline{M}. \quad (5)$$

If $Q_t = \overline{M}$, set $\beta_{-i,t}^{-(j)}(\mathbf{b}) = \epsilon, \forall j \in [K]$. We illustrate an example of such a bid history in Table 1.

Note that for any (universally) feasible bidding policy, the maximum number of units that the bidder can get allocated in round t is Q_t . We claim that $\mathbf{b} = \langle (\frac{w_{Q_1}}{1+\gamma}, q_1), \dots, (\frac{w_{Q_k}}{1+\gamma}, q_k) \rangle$ is an optimal bidding policy for $\mathcal{B}_{-i}(\mathbf{b})$.

To see this, consider any round t . In this round, exactly Q_t bids, namely $\mathbf{b}[1 : t]$, are higher than the competing bids in $\mathcal{B}_{-i}(\mathbf{b})$. As $\mathbf{b} \in \overline{\mathcal{P}}_m$, it is UF. Hence, bidding $\mathbf{b} = \langle (\frac{w_{Q_1}}{1+\gamma}, q_1), \dots, (\frac{w_{Q_k}}{1+\gamma}, q_k) \rangle$ wins Q_t units in round $t, \forall t \in [k]$. So, it is an optimal bidding policy.

The proof of the second property is then completed by the following lemma.

Lemma 2. *For any nested k -uniform bidding policy $\mathbf{b} \in \overline{\mathcal{P}}_m$, let $\mathcal{B}_{-i}(\mathbf{b})$ be the constructed competing bids, presented in Eq. (5), then the value obtained under \mathbf{b} and $\mathcal{B}_{-i}(\mathbf{b}) = [\beta_{-i}^t(\mathbf{b})]_{t \in [T]}$ is strictly larger than that under ANY k' -uniform bid $\mathbf{b}' \in \overline{\mathcal{P}}_m$ and $\mathcal{B}_{-i}(\mathbf{b})$ where $k' \leq k$. That is, $\sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t(\mathbf{b})) > \sum_{t=1}^T V_i(\mathbf{b}'; \beta_{-i}^t(\mathbf{b})), \forall \mathbf{b}' \in \overline{\mathcal{P}}_m \setminus \{\mathbf{b}\}$.*

We present the proof of Lemma 2 in Appendix A.3. The proof primarily uses the result from Lemma 3 which warrants an in-depth discussion and hence presented separately.

Table 1: Suppose $\overline{M} = 6, m = 3, \gamma = 0$ and the average cumulative vector is $\mathbf{w} = [w_1, w_2, \dots, w_6]$. Consider the 3-uniform bid: $\mathbf{b} = \langle (w_2, 2), (w_3, 1), (w_5, 2) \rangle$. So, $Q_1 = 2, Q_2 = 3$ and $Q_3 = 5$. The constructed bid history $\mathcal{B}_{-i}(\mathbf{b})$, presented in Eq. (5), is depicted below, where $C \gg w_1$ and $\epsilon > 0$ is a sufficiently small real number.

$t = 1$	$t = 2$	$t = 3$
C	C	C
C	C	$w_6 + \epsilon$
C	C	$w_6 + \epsilon$
C	$w_4 + \epsilon$	$w_6 + \epsilon$
$w_3 + \epsilon$	$w_4 + \epsilon$	$w_6 + \epsilon$
$w_3 + \epsilon$	$w_4 + \epsilon$	$w_6 + \epsilon$

5 Price of Universal Feasibility

In this section, we quantify the inefficiency caused by restricting the bidding policies to the optimal UF class using the price of universal feasibility (PoUF) metric. Recall from Section 2.4 that POUF_m is the ratio of the value obtained by an optimal feasible bidding policy for a given \mathcal{B}_{-i} and the value obtained by an optimal *universally* feasible bidding policy in the worst case scenario.

We warm up by considering the case when the valuations are homogeneous (i.e., $v_i = v$ for all $i \in [\overline{M}]$). The assumption that bidders have equal valuations for items in multi-unit auctions is standard in several papers in literature (Brânzei et al., 2023b; Dobzinski et al., 2012; Borgs et al., 2005). This setting is alternatively termed as *linear* multi-unit auctions as the total value obtained is a linear function of the number of units obtained (Brânzei et al., 2023b).

Theorem 5.1. *If the bidders have homogeneous valuations for the units (i.e., $v_i = v, \forall i \in [\overline{M}]$), then, $\forall m \in \mathbb{Z}_{>0}$, $\text{POUF}_m = 1$. In other words, enforcing the bidding policies to be UF is lossless.*

The proof is presented in Appendix A.4. If $\gamma = 0$, Theorem 5.1 implies that in the homogeneous valuations setting, bidding truthfully is a weakly dominated strategy for value maximizing bidders, reinforcing the notion of simplified bidding strategies in uniform price auctions in this case.

We now present the main result of this section:

Theorem 5.2 (Price of Universal Feasibility). *For any $m \in \mathbb{Z}_{>0}$, $\text{POUF}_m = \max_{\mathcal{B}_{-i}} \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_m^*(\mathcal{B}_{-i})} \leq 2 - \theta$, where $\theta \in (0, 1]$ and the bound is tight. In other words,*

- for all possible bid histories $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$, there exists a nested k -uniform bidding policy $\mathbf{b} \in \mathcal{P}_m^*$, where $k \in [m]$, which attains at least half of the total value obtained by the optimal bidding policy which is not necessarily UF. That is, $\sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t(\mathbf{b})) \geq 1/2 \cdot V_m^{\text{OPT}}(\mathcal{B}_{-i})$.
- Furthermore, there exists a bid history $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$ and valuation curve (equivalently \mathcal{P}_m^*) such that, for the optimal policy in \mathcal{P}_m^* , the bound is tight. That is, for any choice of $\delta \in (0, 1/2]$, $\sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t(\mathbf{b})) \leq 1/(2 - \delta) \cdot V_m^{\text{OPT}}(\mathcal{B}_{-i})$.

Theorem 5.2 implies that restricting the bidding policy to have the appealing property of being UF has a bounded price and does not lead to an arbitrary loss in the obtained value. Additionally, the factor of two loss under UF policies represents a worst-case scenario, and the setting in which it is attained is quite non-trivial (see Appendix B.2). Thus, we expect the UF policies to be near-optimal in practice, which is also corroborated by the numerical simulations in Section 8.

5.1 Proof of Theorem 5.2: Price of Universal Feasibility

Here, we present only the proof of the upper bound on POUF_m . The tightness of the bound is discussed in Appendix B.1 (for $m = 1$) and Appendix B.2 (for $m \geq 2$).

We begin by defining the following metric for any bid history, $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$,

$$\text{POUF}_m(\mathcal{B}_{-i}) := \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_m^*(\mathcal{B}_{-i})} = \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{\max_{\mathbf{b} \in \mathcal{P}_m^*} \sum_{t=1}^T V_i(\mathbf{b}, \beta_{-i}^t)}. \quad (6)$$

By definition, $\text{POUF}_m = \max_{\mathcal{B}_{-i}} \text{POUF}_m(\mathcal{B}_{-i})$. We first establish upper bounds on $\text{POUF}_m(\mathcal{B}_{-i})$ and then maximizing over \mathcal{B}_{-i} , we complete the proof.

Without loss of generality, let $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i}) = \langle (b_1^*, q_1^*), \dots, (b_m^*, q_m^*) \rangle$ be the optimal solution to Problem (VM- OPT_m) for $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$ and define $Q_\ell^* = \sum_{j \leq \ell} q_j^*$. Suppose $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ is allocated r_t^* units in round t . For any $j \in [m]$, let T_j be the set of rounds in which the least winning bid is b_j^* , i.e.,

$$T_j = \{t \in [T] : Q_{j-1}^* < r_t^* \leq Q_j^*\}. \quad (7)$$

For any $j \in [m]$, partition T_j into $T_{j,0}$ and $T_{j,1}$ such that $T_{j,0}$ is the set of rounds where the bidder gets *strictly less than* Q_j^* units and $T_{j,1}$ is the set of rounds when she gets *exactly* Q_j^* units:

$$T_{j,0} = \{t \in [T_j] : r_t^* < Q_j^*\}, \quad T_{j,1} = \{t \in [T_j] : r_t^* = Q_j^*\}. \quad (8)$$

For any $j \in [m]$, let

$$\widehat{Q}_j = \begin{cases} \max\{r_t^* : t \in T_{j,0}\}, & \text{if } T_{j,0} \neq \emptyset \\ Q_j^* & \text{if } T_{j,0} = \emptyset. \end{cases}$$

In other words, \widehat{Q}_j is the 2^{nd} highest number of units allocated to $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ over the rounds in T_j . Then, the result is obtained by showing the following claim.

Claim 1 (Upper Bound on $\text{POUF}_m(\mathcal{B}_{-i})$). *For any \mathcal{B}_{-i} , $\text{POUF}_m(\mathcal{B}_{-i}) \leq 2 - \theta_{\mathcal{B}_{-i}}$, where*

$$0 < \theta_{\mathcal{B}_{-i}} \leq \bar{\theta}_{\mathcal{B}_{-i}} := \max_{j \in [m]} \frac{W_{\widehat{Q}_j}}{W_{Q_j^*}} \leq 1$$

Here, $\mathbf{v} = [1, v_2, \dots, v_M]$ and $W_j = 1 + v_2 + \dots + v_j$. The exact characterization of $\theta_{\mathcal{B}_{-i}}$ is presented in Appendix A.7. The upper bound, $\bar{\theta}_{\mathcal{B}_{-i}}$ is the maximum ratio of the value obtained from the first \widehat{Q}_j units to that from the first Q_j^* units, where the maximum is taken over all $j \in [m]$.

So, maximizing $\text{POUF}_m(\mathcal{B}_{-i})$ (equivalently minimizing $\theta_{\mathcal{B}_{-i}}$) over all \mathcal{B}_{-i} , we get that $\text{POUF}_m \leq 2 - \theta$ for $\theta = \min_{\mathcal{B}_{-i}} \theta_{\mathcal{B}_{-i}} \in (0, 1]$, as desired.

Constructing a Restricted Class of UF Policies. Here, as a crucial part of the proof, we construct a restricted class of UF policies, denoted by $\mathcal{P}_m^*(\mathcal{B}_{-i})$, where $\mathcal{P}_m^*(\mathcal{B}_{-i}) \subset \mathcal{P}_m^*$. This construction serves two purposes. First, it reduces the search space for the optimal UF bidding policy. Second, and more importantly, it enables us to establish a connection between the optimal UF bidding policy and $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$. In defining the restricted class, we use the quantities, $\{\widehat{Q}_j, Q_j^*\}_{j \in [m]}$ as follows:

$$\mathcal{P}_m^*(\mathcal{B}_{-i}) = \left\{ \mathbf{b} = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle : b_\ell = \frac{w_{Q_\ell}}{1 + \gamma}, \quad Q_\ell \in \{\widehat{Q}_\ell, Q_\ell^*\}, \quad \forall \ell \in [m] \right\}. \quad (9)$$

Recall that any policy in \mathcal{P}_m^* with m bid-quantity pairs takes the form of $\mathbf{b} = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$: $b_\ell = \frac{w_{Q_\ell}}{1+\gamma}, \forall \ell \in [m]$. The policies in $\mathcal{P}_m^*(\mathcal{B}_{-i})$ also have the same structure but as an important difference, for any $\ell \in [m]$, we enforce $Q_\ell \in \{\widehat{Q}_\ell, Q_\ell^*\}$. Observe that for any $\ell \in [m-1]$, $Q_\ell \leq Q_\ell^* < \widehat{Q}_{\ell+1} \leq Q_{\ell+1}$, where the first and third inequalities follow directly from the definition of Q_ℓ and the second one follows from the definition of $\widehat{Q}_{\ell+1}$. So, Q_ℓ 's are distinct and ordered. Further observe that the number of bidding policies in $\mathcal{P}_m^*(\mathcal{B}_{-i})$ is $O(2^m)$; significantly smaller than the number of policies in \mathcal{P}_m^* , which is $O(\overline{M}^m)$.

With the definition of the restricted class of UF policies, we have

$$\text{POUF}_m(\mathcal{B}_{-i}) = \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{\max_{\mathbf{b} \in \mathcal{P}_m^*} \sum_{t=1}^T V_i(\mathbf{b}, \boldsymbol{\beta}_{-i}^t)} \leq \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{\max_{\mathbf{b} \in \mathcal{P}_m^*(\mathcal{B}_{-i})} \sum_{t=1}^T V_i(\mathbf{b}, \boldsymbol{\beta}_{-i}^t)}. \quad (10)$$

Value under the Optimal (Restricted) UF Policies. Here, we characterize the optimal value under the optimal (restricted) UF policies; that is, $\max_{\mathbf{b} \in \mathcal{P}_m^*(\mathcal{B}_{-i})} \sum_{t=1}^T V_i(\mathbf{b}, \boldsymbol{\beta}_{-i}^t)$. To do so, we establish an important relation between m -uniform and 1-uniform bidding policies, which is of independent interest and is invoked several times in the paper. For a bid history $\mathcal{B}_{-i} = [\boldsymbol{\beta}_{-i}^t]_{t \in [T]}$, let $V_i((b, q); \boldsymbol{\beta}_{-i}^t)$ denote the value obtained under the 1-uniform bid (b, q) in round t .

Lemma 3. *For any $m \in \mathbb{Z}_{>0}$, let $\mathbf{b} = \langle \left(\frac{w_{Q_1}}{1+\gamma}, q_1\right), \dots, \left(\frac{w_{Q_m}}{1+\gamma}, q_m\right) \rangle$ be a m -uniform UF bid and $Q_\ell = \sum_{j=1}^\ell q_j$. Then,*

$$V_i(\mathbf{b}; \boldsymbol{\beta}_{-i}^t) = \max_{\ell \in [m]} V_i\left(\left(\frac{w_{Q_\ell}}{1+\gamma}, Q_\ell\right), \boldsymbol{\beta}_{-i}^t\right), \quad \forall t \in [T].$$

We state and prove a stronger version of Lemma 3 in Appendix A.5. By Lemma 3, for any $\mathbf{b} \in \mathcal{P}_m^*(\mathcal{B}_{-i})$, we can express $V_i(\mathbf{b}, \boldsymbol{\beta}_{-i}^t)$ as the maximum of the value obtained by m 1-uniform UF bidding policies.

Let $V_{j,k}$ be the time-average value obtained by $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ in the set of rounds in $T_{j,k}$ (as defined in Eq. (8)). Formally, $\forall j \in [m], k \in \{0, 1\}$,

$$V_{j,k} = \frac{1}{|T_{j,k}|} \sum_{t \in T_{j,k}} V_i(\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i}), \boldsymbol{\beta}_{-i}^t).$$

Note that $V_{j,1} = W_{Q_j^*}$ as $V_{j,1} = 1 + v_2 + \dots + v_{Q_j^*}$ because in any $t \in T_{j,1}$, we have $r_t^* = Q_j^*$. Now, fix some $\mathbf{b} \in \mathcal{P}_m^*(\mathcal{B}_{-i})$, where $\mathbf{b} = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$: $b_\ell = \frac{w_{Q_\ell}}{1+\gamma}$, and $Q_\ell \in \{Q_\ell^*, \widehat{Q}_\ell\}$ for any $\ell \in [m]$. Given this bidding policy, define $N^*, \widehat{N} \subseteq [m]$ as

$$N^* = \{j : Q_j = Q_j^*, j \in [m]\} \quad \text{and} \quad \widehat{N} = \{j : Q_j = \widehat{Q}_j, j \in [m]\}.$$

That is, N^* (respectively, \widehat{N}) are the set of indices in $\mathbf{b} \in \mathcal{P}_m^*(\mathcal{B}_{-i})$ under which we set Q_j to Q_j^* (respectively, \widehat{Q}_j).

With these definitions, we are ready to express the value under $\mathbf{b} \in \mathcal{P}_m^*(\mathcal{B}_{-i})$. By Lemma 3, for any round $t \in T_j$, the value of $\mathbf{b} \in \mathcal{P}_m^*(\mathcal{B}_{-i})$ can be written as the maximum of m 1-uniform bids $\left(\frac{w_{Q_\ell}}{1+\gamma}, Q_\ell\right)$, $\ell \in [m]$:

$$V_i(\mathbf{b}, \boldsymbol{\beta}_{-i}^t) = \max_{\ell \in [m]} V_i\left(\left(\frac{w_{Q_\ell}}{1+\gamma}, Q_\ell\right), \boldsymbol{\beta}_{-i}^t\right) \geq V_i\left(\left(\frac{w_{Q_j}}{1+\gamma}, Q_j\right), \boldsymbol{\beta}_{-i}^t\right), \quad (11)$$

$$\implies \sum_{t=1}^T V_i(\mathbf{b}, \boldsymbol{\beta}_{-i}^t) \geq \sum_{j \in N^*} \sum_{t \in T_j} V_i\left(\left(\frac{w_{Q_j}}{1+\gamma}, Q_j\right), \boldsymbol{\beta}_{-i}^t\right) + \sum_{j \in \widehat{N}} \sum_{t \in T_j} V_i\left(\left(\frac{w_{Q_j}}{1+\gamma}, Q_j\right), \boldsymbol{\beta}_{-i}^t\right). \quad (12)$$

We then invoke the following lemma to compute $\sum_{t \in T_j} V_i\left(\left(\frac{w_{Q_j}}{1+\gamma}, Q_j\right), \boldsymbol{\beta}_{-i}^t\right)$ for any $j \in [m]$.

Lemma 4. *Let $\mathbf{b} \in \mathcal{P}_m^*(\mathcal{B}_{-i})$, where $\mathbf{b} = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle : b_j = \frac{w_{Q_j}}{1+\gamma}$, and $Q_j \in \{Q_j^*, \widehat{Q}_j\}$ for any $j \in [m]$. For any $j \in [m]$,*

- if $Q_j = \widehat{Q}_j$ (i.e., $j \in \widehat{N}$), we have

$$\sum_{t \in T_j} V_i\left(\left(\frac{w_{Q_j}}{1+\gamma}, Q_j\right), \boldsymbol{\beta}_{-i}^t\right) \geq V_{j,0}|T_{j,0}| + W_{\widehat{Q}_j}|T_{j,1}|. \quad (13)$$

- If $Q_j = Q_j^*$ (i.e., $j \in N^*$), we have

$$\sum_{t \in T_j} V_i\left(\left(\frac{w_{Q_j}}{1+\gamma}, Q_j\right), \boldsymbol{\beta}_{-i}^t\right) \geq W_{Q_j^*}|T_{j,1}|. \quad (14)$$

The proof of Lemma 4 is presented in Appendix A.6. Note that the lower bound in Eq. (12) depends only on the choice of the partitions N^* and \widehat{N} . Substituting the lower bound from Lemma 4 in (10), we establish that for any (N^*, \widehat{N}) ,

$$\begin{aligned} \text{POUF}_m(\mathcal{B}_{-i}) &\leq \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{\max_{\mathbf{b} \in \mathcal{P}_m^*(\mathcal{B}_{-i})} \sum_{t=1}^T V_i(\mathbf{b}; \boldsymbol{\beta}_{-i}^t)} \\ &\leq \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{\sum_{j \in N^*} (W_{Q_j^*}|T_{j,1}|) + \sum_{j \in \widehat{N}} (V_{j,0}|T_{j,0}| + W_{\widehat{Q}_j}|T_{j,1}|)}. \end{aligned} \quad (15)$$

To complete the claim, we establish an upper bound on (15) (equivalently on $\text{POUF}_m(\mathcal{B}_{-i})$) by:

Lemma 5. *For any \mathcal{B}_{-i} , let N_0^*, \widehat{N}_0 be the partition of $[m]$ that minimizes RHS in Eq. (15). Then,*

$$\frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{\sum_{j \in N_0^*} (W_{Q_j^*}|T_{j,1}|) + \sum_{j \in \widehat{N}_0} (V_{j,0}|T_{j,0}| + W_{\widehat{Q}_j}|T_{j,1}|)} \leq 2 - \theta_{\mathcal{B}_{-i}},$$

where $0 < \theta_{\mathcal{B}_{-i}} \leq \max_{j \in [m]} \frac{W_{\widehat{Q}_j}}{W_{Q_j^*}} \leq 1$. So, $\text{POUF}_m(\mathcal{B}_{-i}) \leq 2 - \theta_{\mathcal{B}_{-i}}$.

The proof of Lemma 5 is presented in Appendix A.7.

6 Increasing the Number of Bid Quantity Pairs

In this section, we study the impact of increasing the number of bid-quantity pairs on the total value obtained compared to the base case when $m = 1$. Intuitively, a m -uniform policy is more likely to get a higher value than a 1-uniform policy. We formalize this notion in this section and prove that the increase is linear in m . Recall that in Section 2.5, we defined two metrics: $\text{RATIO-UNI}_m := \max_{\mathcal{B}_{-i}} \frac{V_m^*(\mathcal{B}_{-i})}{V_1^*(\mathcal{B}_{-i})}$ and $\text{RATIO-OPT}_m := \max_{\mathcal{B}_{-i}} \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_1^{\text{OPT}}(\mathcal{B}_{-i})}$. We now present the main result of this section.

Theorem 6.1 (Impact of increasing m). *For any $m \geq 2$, $1 \leq \text{RATIO-UNI}_m, \text{RATIO-OPT}_m < m$.*

- *In other words, for any given bid history, the optimal (UF) 1-uniform bidding policy obtains at least $(1/m)^{\text{th}}$ of the value obtained by the optimal (UF) k -uniform bidding policy, where $k \in [m]$.*
- *Furthermore, the upper bound is tight, i.e., for any $\delta \in (0, 1/2]$, there exists a bid history, \mathcal{B}_{-i} and a valuation curve such that $V_1^*(\mathcal{B}_{-i}) = 1/(m-\delta) \cdot V_m^*(\mathcal{B}_{-i})$ and $V_1^{\text{OPT}}(\mathcal{B}_{-i}) = 1/(m-\delta) \cdot V_m^{\text{OPT}}(\mathcal{B}_{-i})$.*

The proof of the upper bound is presented in Appendix A.8 and the tightness of the bound is discussed in Appendix B.3.

We finally present a result relating the inefficiency caused by restricting the policies to be UF (Theorem 5.2) and that caused by restricting to only 1-uniform bidding policies (Theorem 6.1).

Theorem 6.2. *For any $m \in \mathbb{Z}_{>0}$, $1 \leq \max_{\mathcal{B}_{-i}} \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_1^*(\mathcal{B}_{-i})} < 2m$.*

- *Put differently, for any bid history, the value obtained by the optimal 1-uniform UF bidding policy is at least $(1/2m)^{\text{th}}$ of the value obtained by the optimal k -uniform bidding policy, where $k \in [m]$, which is not necessarily UF.*
- *Moreover, the upper bound is tight, that is, for any $\delta \in (0, 1/2]$, there exists a \mathcal{B}_{-i} and valuation curve such that $V_1^*(\mathcal{B}_{-i}) = 1/(2m - \delta) \cdot V_m^{\text{OPT}}(\mathcal{B}_{-i})$.*

The lower bound holds trivially as $V_m^{\text{OPT}}(\mathcal{B}_{-i}) \geq V_1^{\text{OPT}}(\mathcal{B}_{-i}) \geq V_1^*(\mathcal{B}_{-i})$ for any \mathcal{B}_{-i} . To obtain the upper bound, observe that by Theorem 5.2 for $m = 1$ and Theorem 6.1, we get

$$\max_{\mathcal{B}_{-i}} \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_1^*(\mathcal{B}_{-i})} \leq \max_{\mathcal{B}_{-i}} \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_1^{\text{OPT}}(\mathcal{B}_{-i})} \cdot \max_{\mathcal{B}_{-i}} \frac{V_1^{\text{OPT}}(\mathcal{B}_{-i})}{V_1^*(\mathcal{B}_{-i})} = \text{RATIO-OPT}_m \cdot \text{POUF}_1 < 2m.$$

The tightness of the bound is presented in Appendix B.4. To understand the implication of the theorem, recall that we introduce inefficiency due to two factors: (i) restricting the bids to be UF and (ii) choosing the number of bid-quantity pairs in the bidding policy. Theorem 6.2 (specifically the tightness of the upper bound) suggests that the effects of both the inefficiencies are decoupled.

7 Identifying the Optimal Universally Feasible Policy

In the previous sections, we characterized the optimal class of m -uniform UF bidding policies and showed that the class has a good performance while adhering to UF property. In this section, we discuss the problem of learning the optimal UF policy on the fly as the bidder participates in auctions over time.

Formally, consider a repeated setting where in every round t , the seller announces the auction for K identical items. The bidder submits a bid $\mathbf{b}^t \in \mathcal{P}_m^*$ with no knowledge about the bids submitted by the competitors for round t . The seller collects the bids from all the bidders, β_{-i}^t , and allocates the items to the bidders with the top K bids by setting the K^{th} highest bid as the clearing price.

As before, we consider the problem from the perspective of a single RoI-constrained, value-maximizing bidder. Our goal is then to design no-regret (sublinear regret) learning algorithms for bidding, where the regret is the difference between the value under the optimal UF policy in \mathcal{P}_m^* and what the bidder earns:

$$\text{REGRET}(T) = \max_{\mathbf{b} \in \mathcal{P}_m^*} \sum_{t=1}^T V_i(\mathbf{b}, \beta_{-i}^t) - \sum_{t=1}^T \mathbb{E} [V_i(\mathbf{b}^t, \beta_{-i}^t)],$$

where the expectation is with respect to any randomness in the learning algorithm. To design a no-regret learning policy, when the number of bid-quantity pairs m is small, which is typically the case in practice, one can leverage existing learning algorithms such as Hedge (Freund and Schapire, 1997) (for the full information setting)⁴ and the EXP3 algorithm (Auer et al., 1995) (for the bandit setting), leading to a regret of $\text{REGRET}(T) = \tilde{O}(\bar{M}\sqrt{T})$ and $\tilde{O}(\bar{M}^{1+0.5m}\sqrt{T})$, respectively, where $\tilde{O}(\cdot)$ hides the logarithmic factors. Hence, from the results of Theorem 6.1, we observe that increasing the number of bids from 1 to any general m can increase the total value obtained by at most a factor linear in m but the number of rounds required to identify such a policy can grow exponentially in m . Each bidding policy in \mathcal{P}_m^* can be viewed as an expert in the Hedge and EXP3 algorithm. As m is a constant, the number of experts, which is $O(\bar{M}^m)$, is also small.

For the case when m is large, the aforementioned learning algorithms are not practically appealing. Nevertheless, we design a method to obtain an approximately optimal nested m -uniform bidding policy. The design of this algorithm is based on the following direct corollary of Lemma 3:

Corollary 7.1. *For any $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$, we have*

$$\max_{\mathbf{b} \in \mathcal{P}_m^*} \sum_{t=1}^T V_i(\mathbf{b}, \beta_{-i}^t) = \max_{S \subseteq [\bar{M}]: |S| \leq m} \sum_{t=1}^T \max_{\ell \in S} V_i \left(\left(\frac{w_\ell}{1+\gamma}, \ell \right), \beta_{-i}^t \right).$$

The proof of Corollary 7.1 is presented in Appendix A.9. This corollary shows that the offline problem of identifying the optimal UF bidding policy can be viewed as maximizing a certain monotone submodular function of the form: $f_t(S) = \max_{\ell \in S} V_i \left(\left(\frac{w_\ell}{1+\gamma}, \ell \right), \beta_{-i}^t \right)$, subject to cardinality constraints. The submodularity of f_t follows directly from Cornuéjols et al. (1983, Theorem 9.1). With this observation, one can employ algorithms for online submodular optimization under cardinality constraint to obtain sublinear $(1 - 1/e)$ -approximate regret of the order $\tilde{O}(m\bar{M}\sqrt{T})$ in the full information setting and $\tilde{O}(m\bar{M}^{5/3}T^{2/3})$ in the bandit setting (Niazadeh et al., 2022).

8 Numerical Simulations

In this section, we conduct numerical experiments to study the results of this work in practice. Specifically, we investigate two key metrics that measure the performance of UF bidding policies: $\frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_m^*(\mathcal{B}_{-i})}$ (proxy for POUF_m) and $\frac{V_m^*(\mathcal{B}_{-i})}{V_1^*(\mathcal{B}_{-i})}$ (proxy for RATIO-UNI_m) as a function of m . Recall that, $\text{POUF}_m < 2$, irrespective of m (Theorem 5.2), and $\text{RATIO-UNI}_m < m$ (Theorem 6.1).

Dataset and Parameters. We conduct our simulations for the EU ETS emission permit auction data for 2022 and 2023. However, only aggregate statistics of the submitted bids is publicly available for privacy reasons. Hence, we synthesize individual level bid data from these available statistics. The exact procedure to reconstruct the bid data is presented in Appendix C.1. The bids are normalised to be in $[0, 1]$. We sample the values from the $\text{Unif}[0, 1]$ distribution. In each simulation, we sample $T \sim \text{Unif}[100, 300]$ auctions and let $\bar{M} \sim \text{Unif}[10, 80]$. We solve (VM-UNI $_m$) by formulating it as an integer linear program (ILP) by Corollary 7.1. As computing (VM-OPT $_m$) can be non-trivial, we obtain an uniform upper bound for $V_m^{\text{OPT}}(\mathcal{B}_{-i})$. The ILP and the uniform upper bound for $V_m^{\text{OPT}}(\mathcal{B}_{-i})$ is presented in Appendix C.2. We vary $m = 1$ to $m = 10$ and average over 100 simulations. The results are plotted in Fig. 4.

Price of Universal Feasibility. The simulations (i.e., the left plot of Fig. 4) show that (the upper bound for) $\frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_m^*(\mathcal{B}_{-i})}$ rapidly decays to 1 with increase in m . Specifically, for $m \geq 4$, the

⁴In the full information setting, based on the received feedback, the bidder can compute $V_i(\mathbf{b}, \beta_{-i}^t)$ for all $\mathbf{b} \in \mathcal{P}_m^*$. In the bandit setting, the bidder gets information about her own allocation, i.e., $V_i(\mathbf{b}^t, \beta_{-i}^t)$ only.

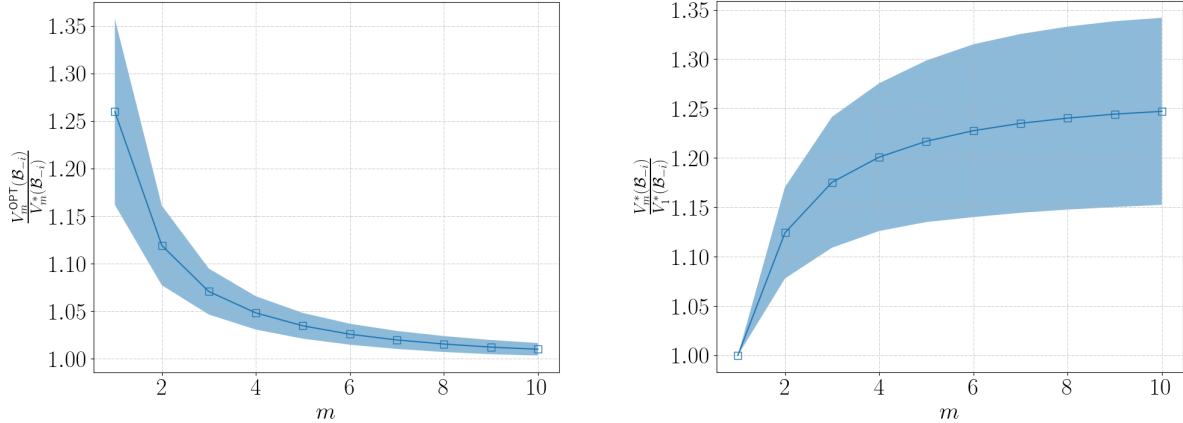


Figure 4: The left (resp. right) figure corresponds to $\frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_m^*(\mathcal{B}_{-i})}$ (resp. $\frac{V_m^*(\mathcal{B}_{-i})}{V_1^*(\mathcal{B}_{-i})}$) as a function of m . The shaded area denotes the uncertainty of one standard deviation on either side.

ratio of the value obtained by the optimal bidding policy (not necessarily UF) to that by UF bidding policies is ~ 1.05 indicating that the UF policies are near-optimal. Interestingly, this threshold value of m aligns with the average number of bid-quantity pairs submitted by bidders in the actual EU ETS auctions during this time frame (4.35 for 2023 and 3.89 for 2022) (EEX, 2023).

Increasing the Number of Bid Quantity Pairs. From the right plot of Fig. 4, we observe that the empirical values of $\frac{V_m^*(\mathcal{B}_{-i})}{V_1^*(\mathcal{B}_{-i})}$ is significantly better than the worst-case bound m (Theorem 6.1). In fact, the gain obtained by increasing the number of bid-quantity pairs plateaus after $m = 4$ and even for $m = 10$, the ratio of the value obtained by a 10-uniform bidding policy to a classical (1-)uniform bidding policy is ~ 1.25 . Both the results indicate that UF bidding policies with small values of m are practically optimal.

9 Conclusion and Open Problems

We studied uniform price auctions, widely used in various markets. While these auctions offer fair pricing, bidder behavior may deviate from theoretical predictions. We examine how bidders, seeking to maximize value while meeting RoI constraints, behave in these auctions. Our study characterizes the optimal bidding policies that maintain universal RoI feasibility regardless of competitors’ bids, showing that this class depends solely on the bidders’ valuations and RoI parameters. We also quantify the cost of enforcing the universal RoI feasibility property and explore the benefits of increasing the number of bid-quantity pairs. Additionally, we discussed methods for identifying optimal bidding policies and address challenges in scenarios with a large number of bid-quantity pairs. Furthermore, we conducted numerical simulations that show the universally feasible bidding policies perform substantially better than the theoretical results presented in this study.

Our study opens the door to several intriguing areas of interest for further exploration. First, analyzing discriminatory price auctions, also known as “pay-as-bid” auctions, could shed light on bidding strategies under this alternative auction framework. Second, introducing both RoI and budgetary constraints would offer a more realistic model of bidder behavior in practical scenarios where bidders may have financial limitations beyond just RoI considerations. Lastly, the extension of our findings to scenarios involving non-identical items, such as those encountered in combinatorial auctions, might provide valuable insights into more complex auction environments where items possess diverse characteristics and values.

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A Omitted Proofs

A.1 Proof of Theorem 3.1

Define the following class of no-overbidding (NOB) bidding policies containing upto m bid-quantity pairs:

$$\mathcal{P}_m^{\text{NOB}} := \left\{ \mathbf{b} = \langle (b_1, q_1), \dots, (b_k, q_k) \rangle : b_\ell \leq \frac{w_{Q_\ell}}{1+\gamma}, \forall \ell \in [k], \forall k \in [m] \right\}.$$

We first show that $\mathcal{P}_m \subseteq \mathcal{P}_m^{\text{NOB}}$, and then complete the proof by showing $\mathcal{P}_m^{\text{NOB}} \subseteq \mathcal{P}_m$.

Proof of $\mathcal{P}_m \subseteq \mathcal{P}_m^{\text{NOB}}$. We start by presenting a lemma about the bidder's per unit payments, which we will also use to prove several other results.

Lemma 6 (Per-unit Payments). *Suppose the bidder bids $\mathbf{b} = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$. Recall that $Q_j = \sum_{\ell=1}^j q_\ell, \forall j \in [m]$. Recall that, $\beta_{-i,t}^{-(j)}$ is the j^{th} smallest winning bid in the absence of bids from bidder i for round t . If $j = 0$, $\beta_{-i,t}^{-(j)} = 0$ and $j \geq K$, $\beta_{-i,t}^{-(j)} = \infty$. Then, the per-unit payments by the bidder in round t is*

$$p(\beta^t) = \begin{cases} 0, & \text{if } x_i(\beta^t) = 0 \\ b_\ell, & \text{if } Q_{\ell-1} < x_i(\beta^t) < Q_\ell \\ \min(b_\ell, \beta_{-i,t}^{-(Q_\ell+1)}), & \text{if } x_i(\beta^t) = Q_\ell \end{cases}. \quad (16)$$

The proof of Lemma 6 is presented in Appendix A.1.1.

Observation 1. *Overbidding is not an universally feasible bidding policy. To see this, let $\mathbf{b} = \langle (b_1, q_1), \dots, (b_\ell, q_\ell), \dots, (b_m, q_m) \rangle$ be an overbid such that $b_\ell > w_{Q_\ell}/(1+\gamma)$. Now, consider a bid history with a single round (i.e., $T = 1$) in which the individual bids are:*

$$\beta_{-i,1}^{-(j)} = \begin{cases} \epsilon, & \text{if } 1 \leq j \leq Q_\ell \\ C, & \text{if } Q_\ell < j \leq K \end{cases},$$

where $\epsilon \sim 0$ and $C \gg w_1$. Submitting \mathbf{b} gets allocated Q_ℓ units and from Lemma 6, we conclude that the clearing price of auction is b_ℓ . As $b_\ell > w_{Q_\ell}/(1+\gamma)$ by assumption, the RoI constraint is violated.

As overbidding is not UF, every UF bidding policy is a NOB bidding policy, i.e., $\mathcal{P}_m \subseteq \mathcal{P}_m^{\text{NOB}}$.

Proof of $\mathcal{P}_m^{\text{NOB}} \subseteq \mathcal{P}_m$. We now prove that the converse is also true, i.e., every NOB bidding policy is also UF. To show this, fix any arbitrary bid history, $\mathcal{B}_{-i} = [\beta_{-i,t}^t]_{t \in [T]}$, and consider any $\mathbf{b} = \langle (b_1, q_1), \dots, (b_k, q_k) \rangle \in \mathcal{P}_m^{\text{NOB}}$ where $k \in [m]$. Suppose in round t , bidding \mathbf{b} wins $x_i(\beta^t)$ units. So, from Lemma 6, if $x_i(\beta^t) = 0$, trivially, $P_i(\beta^t) = 0 = V_i(\beta^t)$. If $Q_{\ell-1} < x_i(\beta^t) \leq Q_\ell$, for some ℓ ,

$$P_i(\beta^t) = x_i(\beta^t) \cdot p(\beta^t) \leq x_i(\beta^t) \cdot b_\ell \leq x_i(\beta^t) \cdot \frac{w_{Q_\ell}}{1+\gamma} \leq x_i(\beta^t) \cdot \frac{w_{x_i(\beta^t)}}{1+\gamma} = \frac{V_i(\beta^t)}{1+\gamma}.$$

The first inequality holds true because bidders' per-unit payment is at most their least winning bid (individual rationality of the auction format), the second is true by definition, and the third is true because the w_j is non-decreasing in j and $x_i(\beta^t) \leq Q_\ell$. As the choice of bid history \mathcal{B}_{-i} , round t , and bidding policy \mathbf{b} , was arbitrary, we conclude that every bid in $\mathcal{P}_m^{\text{NOB}}$ is UF, i.e., $\mathcal{P}_m^{\text{NOB}} \subseteq \mathcal{P}_m$, which completes the proof.

A.1.1 Proof of Lemma 6

Consider the following three cases:

1. If $x_i(\beta^t) = 0$, trivially, $p(\beta^t) = 0$.
2. Let $Q_{\ell-1} < x_i(\beta^t) < Q_\ell$. Let b be the last accepted bid after including \mathbf{b} , i.e., $\beta^t = (\mathbf{b}, \beta_{-i}^t)$. Then $b_\ell \stackrel{(a)}{\geq} b \stackrel{(b)}{\geq} b_\ell$.
 - I. (a) holds true because the bidder is allocated at least one unit for bid b_ℓ and
 - II. (b) is correct because she does not acquire at least one unit for bid b_ℓ . Hence, $p(\beta^t) = b_\ell$.
3. Suppose $x_i(\beta^t) = Q_\ell$. If $b_\ell > \beta_{-i,t}^{-(Q_\ell+1)}$, then $p(\beta^t) = \beta_{-i,t}^{-(Q_\ell+1)}$. However, if $b_\ell \leq \beta_{-i,t}^{-(Q_\ell+1)}$, $p(\beta^t) = b_\ell$. So, $p(\beta^t) = \min(b_\ell, \beta_{-i,t}^{-(Q_\ell+1)})$.

A.2 Proof of Lemma 1

Let $\beta^t = (\mathbf{b}, \beta_{-i}^t)$ and $\beta'^t = (\mathbf{b}', \beta_{-i}^t)$. Recall that $\mathbf{b} = [b_1, b_2, \dots, b_k]$ and $\mathbf{b}' = [b'_1, b'_2, \dots, b'_k]$ such that $b_j \geq b'_j, \forall j \in [k]$. Furthermore, by assumption, \mathbf{b} is RoI feasible for $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$.

We first prove that \mathbf{b}' is also feasible for the fixed bid history, \mathcal{B}_{-i} . Contrary to our claim, suppose \mathbf{b}' is infeasible. Then, there must exist a round t in which \mathbf{b}' is allocated r'_t units with clearing price $p(\beta'^t)$ such that the RoI constraint is violated:

$$(1 + \gamma)p(\beta'^t) > w_{r'_t}. \quad (17)$$

For the same round t , suppose \mathbf{b} is allocated r_t units. By definition of allocation rule and payment rule, the units allocated and the clearing price obtained in an auction are weakly increasing in bids, so $r'_t \leq r_t$ and $p(\beta'^t) \leq p(\beta^t)$. As \mathbf{b} is feasible,

$$(1 + \gamma)p(\beta^t) \leq w_{r_t}. \quad (18)$$

Combining Equations (17) and (18), we have

$$(1 + \gamma)p(\beta^t) \leq w_{r_t} \stackrel{(a)}{\leq} w_{r'_t} < (1 + \gamma)p(\beta'^t) \implies p(\beta^t) < p(\beta'^t).$$

which is a contradiction. Here (a) is true as w_j is non-increasing in j and $r'_t \leq r_t$. So, \mathbf{b}' is feasible.

To prove monotonicity, consider any round t . By definition of the allocation rule, the value obtained in an auction is weakly increasing in the bids submitted. As \mathbf{b} and \mathbf{b}' are both feasible, the RoI constraints are valid for both. So, $V_i(\mathbf{b}; \beta_{-i}^t) \geq V_i(\mathbf{b}'; \beta_{-i}^t)$. Summing over all rounds, we get the desired result.

A.3 Proof of Lemma 2

Fix any k -uniform bid $\mathbf{b} \in \overline{\mathcal{P}}_m$. Recall the constructed bid history, $\mathcal{B}_{-i}(\mathbf{b})$, for which \mathbf{b} is an optimal bidding policy in $\overline{\mathcal{P}}_m$. For sake of completeness, we present it again. We constructed a bid history with $T = k$ rounds. In round $t \in [k]$, for a sufficiently small $\epsilon > 0$ and $C \gg w_1$,

$$\beta_{-i,t}^{-(j)}(\mathbf{b}) = \begin{cases} \frac{w_{Q_t+1}}{1+\gamma} + \epsilon, & \text{if } 1 \leq j \leq Q_t \\ C, & \text{if } Q_t < j \leq K \end{cases}, \quad \forall t \in [k].$$

If $Q_t = \bar{M}$, we set $\beta_{-i,t}^{-(j)}(\mathbf{b})$ to ϵ for any $j \in [K]$. We proved that $\mathbf{b} \in \bar{\mathcal{P}}_m$ is an optimal bidding policy for $\mathcal{B}_{-i}(\mathbf{b})$. Furthermore, we also showed that \mathbf{b} obtains the maximum number of units that can be allocated to the bidder in *every* round.

We now show that the value obtained under the k -uniform bid \mathbf{b} and $\mathcal{B}_{-i}(\mathbf{b})$ is strictly larger than that under ANY other k_1 -uniform bid $\mathbf{b}' \in \bar{\mathcal{P}}_m$ and $\mathcal{B}_{-i}(\mathbf{b}')$ where $k_1 \leq k$. That is,

$$\sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t) > \sum_{t=1}^T V_i(\mathbf{b}'; \beta_{-i}^t), \quad \forall \mathbf{b}' \in \bar{\mathcal{P}}_k \setminus \{\mathbf{b}\}.$$

We show the result by contradiction. Contrary to our claim, suppose that there exists an optimal k_1 -uniform bid, $\mathbf{b}' \neq \mathbf{b}$. Let $\mathbf{b}' = \left\langle \left(\frac{w_{Q'_1}}{1+\gamma}, q'_1 \right), \dots, \left(\frac{w_{Q'_{k_1}}}{1+\gamma}, q'_{k_1} \right) \right\rangle \in \bar{\mathcal{P}}_k \setminus \{\mathbf{b}\}$ and define $Q'_j = \sum_{\ell \leq j} q'_\ell$. Let $S = \{Q'_j : j \in [m]\}$. As \mathbf{b}' is UF, invoking Lemma 3, we get that for any round t :

$$V_i(\mathbf{b}'; \beta_{-i}^t) = \max_{\ell \in [k_1]} V_i\left(\left(\frac{w_{Q'_\ell}}{1+\gamma}, Q'_\ell\right); \beta_{-i}^t\right). \quad (19)$$

We show that for each of the k rounds in $\mathcal{B}_{-i}(\mathbf{b}')$, a unique bidding policy of the form: $\left(\frac{w_{Q'_\ell}}{1+\gamma}, Q'_\ell\right)$ maximizes the value obtained for that round. So, if $k_1 < k$, then clearly \mathbf{b}' is suboptimal as there will be at least one round in which \mathbf{b}' will not achieve the same value as \mathbf{b} . So, assume that $k_1 = k$. As the RHS of (19) is a maximum over $k_1 = k$ entities, it can be uniquely identified. Thus, \mathbf{b}' is uniquely determined.

To see this, consider any round t . For a policy $\left(\frac{w_{Q'_\ell}}{1+\gamma}, Q'_\ell\right)$ to win at least 1 unit, it has to be higher than the least winning competing bid. So, $\frac{w_{Q'_\ell}}{1+\gamma} > \frac{w_{Q_t+1}}{1+\gamma} + \epsilon \implies Q'_\ell \leq Q_t$. By assumption, \mathbf{b}' obtains the same value as \mathbf{b} in round t . To win Q_t units in round t , the demand should be for at least Q_t units. So, $Q'_\ell \geq Q_t \implies Q'_\ell = Q_t \implies Q_t \in S$. As t was chosen arbitrarily, $Q_t \in S, \forall t \in [k]$. Now, observe that fixing Q_t 's uniquely determines \mathbf{b}' , implying $\mathbf{b} = \mathbf{b}'$, which is a contradiction. So, the k -uniform bid \mathbf{b} is the unique optimal bidding policy for $\mathcal{B}_{-i}(\mathbf{b})$ in $\mathcal{P}_k^* \subseteq \mathcal{P}_m^*$.

A.4 Proof of Theorem 5.1

Suppose the bidder values each unit equally (i.e., $v_i = v$ for all $i \in [\bar{M}]$). We first compute the optimal UF bidding policy and repeat the exercise by removing the UF condition. We show that the optimal policy in both the cases are identical and thus complete the proof.

Optimal Universally Feasible Bidding Policy. Fix some $m \in \mathbb{Z}_{>0}$. As $v_i = v$ for all $i \in [\bar{M}]$, $w_i = v$ for all $i \in [\bar{M}]$. So,

$$\mathcal{P}_m^* = \left\{ \left(\frac{v}{1+\gamma}, q \right) : q \in [\bar{M}] \right\}. \quad (20)$$

As the bid value remains the same for all $q \in [\bar{M}]$, to maximize value, the bidder must select the policy with highest demand, hence the optimal UF bidding policy is $(v/(1+\gamma), \bar{M})$.

Removing the Universal Feasibility Condition. Here, we characterize the optimal non-UF bidding policy $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ and show that it is $(v/(1+\gamma), \bar{M})$ for any $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$. To do so, we present the following lemma which relates the the number of units allocated to the optimal bidding policy, $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ and 1-uniform UF bidding policies.

Lemma 7. For any \mathcal{B}_{-i} , let $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i}) = \langle (b_1^*, q_1^*), \dots, (b_m^*, q_m^*) \rangle$ be the optimal solution to Problem (VM-OPT_m). Recall $Q_j^* = \sum_{\ell \leq j} q_\ell^*$ where under $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ and \mathcal{B}_{-i} , r_i^* units is allocated to the

bidder in round t , for $t \in [T]$. For any $t \in [T]$, the 1-uniform bid $\left(\frac{w_{r_t^*}}{1+\gamma}, r_t^*\right)$ is allocated exactly r_t^* units.

We state and prove a stronger version of Lemma 7 as Lemma 9 in Appendix A.6.1.

Suppose the bidder is allocated r_t^* units any round $t \in [T]$ under $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ and \mathcal{B}_{-i} . Then, by Lemma 7, the bidding policy $\left(\frac{w_{r_t^*}}{1+\gamma}, r_t^*\right) = \left(\frac{v}{1+\gamma}, r_t^*\right)$ obtains r_t^* units in round t . By Lemma 1 (monotonicity of feasible bids), $\left(\frac{v}{1+\gamma}, \overline{M}\right)$ wins at least r_t^* units in round t . As t was chosen arbitrarily, we conclude that $\left(\frac{v}{1+\gamma}, \overline{M}\right)$ wins at least r_t^* units in every round t implying that $\left(\frac{v}{1+\gamma}, \overline{M}\right)$ obtains at least $V_m^{\text{OPT}}(\mathcal{B}_{-i})$. Trivially, the value obtained by $\left(\frac{v}{1+\gamma}, \overline{M}\right)$ is at most $V_m^{\text{OPT}}(\mathcal{B}_{-i})$. So, bidding $\left(\frac{v}{1+\gamma}, \overline{M}\right)$ obtains $V_m^{\text{OPT}}(\mathcal{B}_{-i})$ for the bid history \mathcal{B}_{-i} . Combining with the results from the previous paragraph, we conclude that $\text{POUF}_m = 1, \forall m \in \mathbb{Z}_{>0}$.

A.5 Proof of Lemma 3 and its Stronger Version (Lemma 8)

To show Lemma 3, we will verify a stronger version of the lemma (Lemma 8 stated below) that has a similar statement, but does not require the bid to be UF.

Lemma 8. Fix a bid history $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$. For \mathcal{B}_{-i} , let $V_i((b, q); \beta_{-i}^t)$ denote the value obtained by bidding (b, q) in round t . For any $m \in \mathbb{Z}_{>0}$, let $\mathbf{b} = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$ be a feasible (not necessarily UF) m -uniform bid for \mathcal{B}_{-i} . Then, we have

$$V_i(\mathbf{b}; \beta_{-i}^t) = \max_{\ell \in [m]} V_i((b_\ell, Q_\ell); \beta_{-i}^t), \forall t \in [T],$$

where we recall that $Q_\ell = \sum_{j=1}^{\ell} q_j$.

Proof. Proof of Lemma 8. We prove the lemma via induction on m . The base case is $m = 1$ for which the result is trivially true. Now assume that the result holds for any m -uniform bid for a fixed bid history $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$. We will then show that the result holds for any $m + 1$ -uniform bid and the same \mathcal{B}_{-i} .

Consider the case for a $m + 1$ -uniform bid $\mathbf{b} = \langle (b_1, q_1), \dots, (b_m, q_m), (b_{m+1}, q_{m+1}) \rangle$. We know by assumption that \mathbf{b} is feasible for \mathcal{B}_{-i} . Suppose that by bidding \mathbf{b} , the bidder is allocated r_t units in round t . There are two cases: (a) $r_t \leq Q_m$ and (b) $r_t > Q_m$.

Case I. $r_t \leq Q_m$. In this case, we have

$$V_i(\mathbf{b}; \beta_{-i}^t) = V_i(\mathbf{b}[1 : m]; \beta_{-i}^t). \quad (21)$$

Claim 2. The bid-quantity pair (b_{m+1}, Q_{m+1}) is feasible for \mathcal{B}_{-i} and for any $t \in [T]$, $V_i(\mathbf{b}; \beta_{-i}^t) \geq V_i((b_{m+1}, Q_{m+1}); \beta_{-i}^t)$.

Proof. Proof of Claim 2. By assumption, \mathbf{b} is feasible, the total demand of \mathbf{b} and (b_{m+1}, Q_{m+1}) are equal and $b_{(j)} \geq b_{m+1}, \forall j \in [Q_{m+1}]$, where $b_{(j)}$ denotes the bid value in j th position in the sorted bid vector. So, by Lemma 1, (b_{m+1}, Q_{m+1}) is feasible and $V_i(\mathbf{b}; \beta_{-i}^t) \geq V_i((b_{m+1}, Q_{m+1}), \beta_{-i}^t)$. \square

Hence, by Claim 2 and Eq. (21),

$$V_i(\mathbf{b}; \beta_{-i}^t) = \max \left\{ V_i(\mathbf{b}[1 : m]; \beta_{-i}^t), V_i((b_{m+1}, Q_{m+1}); \beta_{-i}^t) \right\}. \quad (22)$$

Case II. $r_t > Q_m$. As $r_t > Q_m$, b_{m+1} is the least winning bid which implies $b_{m+1} \geq \beta_{-i,t}^{-(r_t)}$. So, (b_{m+1}, Q_{m+1}) is allocated at least r_t units which implies $V_i((b_{m+1}, Q_{m+1}), \beta_{-i}^t) \geq V_i(\mathbf{b}, \beta_{-i}^t)$. By Claim 2, we also have $V_i(\mathbf{b}, \beta_{-i}^t) \geq V_i((b_{m+1}, Q_{m+1}), \beta_{-i}^t)$. Hence,

$$V_i(\mathbf{b}; \beta_{-i}^t) = V_i((b_{m+1}, Q_{m+1}); \beta_{-i}^t). \quad (23)$$

As $r_t > Q_m$, (b_{m+1}, Q_{m+1}) is allocated at least $Q_m + 1$ units, whereas $\mathbf{b}[1 : m]$ has demand for (hence, can be allocated) at most Q_m units. So,

$$\begin{aligned} V_i(\mathbf{b}; \beta_{-i}^t) &\geq V_i(\mathbf{b}[1 : m]; \beta_{-i}^t) \\ \implies V_i(\mathbf{b}; \beta_{-i}^t) &= \max \left\{ V_i(\mathbf{b}[1 : m]; \beta_{-i}^t), V_i((b_{m+1}, Q_{m+1}); \beta_{-i}^t) \right\}. \end{aligned} \quad (24)$$

For both **Case I** and **Case II**, we get the same result (cf. (22) and (24)). Hence,

$$\begin{aligned} V_i(\mathbf{b}; \beta_{-i}^t) &= \max \left\{ V_i(\mathbf{b}[1 : m]; \beta_{-i}^t), V_i((b_{m+1}, Q_{m+1}); \beta_{-i}^t) \right\} \\ &\stackrel{(a)}{=} \max \left\{ \max_{\ell \in [m]} V_i((b_\ell, Q_\ell); \beta_{-i}^t), V_i((b_{m+1}, Q_{m+1}); \beta_{-i}^t) \right\} \\ &= \max_{\ell \in [m+1]} V_i((b_\ell, Q_\ell); \beta_{-i}^t), \end{aligned}$$

where (a) holds as $\mathbf{b}[1 : \ell]$ is feasible for $\forall \ell \in [m+1]$, and then applying the induction hypothesis on m , we conclude the proof. \square

A.6 Proof of Lemma 4

To prove this result, we use the following key lemma that compares allocations under $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ and 1-uniform UF bidding policies. The proof of this lemma is presented in Appendix A.6.1.

Lemma 9. For any \mathcal{B}_{-i} , let $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i}) = \langle (b_1^*, q_1^*), \dots, (b_m^*, q_m^*) \rangle$ be the optimal solution to Problem (VM-OPT_m). Recall $Q_j^* = \sum_{\ell \leq j} q_\ell^*$ where under $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ and \mathcal{B}_{-i} , r_i^* units is allocated to the bidder in round t , for $t \in [T]$.

1. For any $t \in [T]$ and $q \leq r_i^*$, the 1-uniform bid $\left(\frac{w_q}{1+\gamma}, q\right)$ is allocated exactly q units.
2. For any $j \in [m]$, let $T_j \subseteq [T]$, defined in Eq. (7), be the rounds in which b_j^* is the least winning bid. Suppose that $\exists t \in T_j$ such that $r_i^* < Q_j^*$. Then in any $t' \in T_j$ in which $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ is allocated less than r_i^* units, i.e., $r_{i'}^* \leq r_i^*$, the 1-uniform bidding policy $\left(\frac{w_{r_{i'}^*}}{1+\gamma}, r_{i'}^*\right)$ is allocated at least $r_{i'}^*$ units.

With this lemma, we are ready to show the result.

Case I: $Q_j = \widehat{Q}_j$. By assumption, $Q_j = \widehat{Q}_j \leq Q_j^*$. So, for any $t \in T_{j,1}$, invoking Lemma 9 (1) with $q = \widehat{Q}_j$, we conclude that $\left(\frac{w_{\widehat{Q}_j}}{1+\gamma}, \widehat{Q}_j\right)$ is allocated exactly \widehat{Q}_j units. Summing over all rounds in $T_{j,1}$,

$$\sum_{t \in T_{j,1}} V_i\left(\left(\frac{w_{\widehat{Q}_j}}{1+\gamma}, \widehat{Q}_j\right), \beta_{-i}^t\right) = W_{\widehat{Q}_j} |T_{j,1}|. \quad (25)$$

Now, suppose $T_{j,0} \neq \emptyset$. By definition, $\widehat{Q}_j < Q_j^*$. Also, $\widehat{Q}_j \geq r_s^*$ for any $s \in T_{j,0}$ by definition. So, for any $s \in T_{j,0}$, invoking Lemma 9 (2) with $r_i^* = \widehat{Q}_j$, we conclude that $\left(\frac{w_{\widehat{Q}_j}}{1+\gamma}, \widehat{Q}_j\right)$ is allocated

at least r_s^* units. Hence, summing over all rounds, $\left(\frac{w_{\widehat{Q}_j}}{1+\gamma}, \widehat{Q}_j\right)$ gets at least the value obtained by $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ over the rounds in $T_{j,0}$. So,

$$\sum_{t \in T_{j,0}} V_i\left(\left(\frac{w_{Q_j}}{1+\gamma}, Q_j\right), \beta_{-i}^t\right) \geq \sum_{t \in T_{j,0}} V_i\left(\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i}), \beta_{-i}^t\right) = V_{j,0}|T_{j,0}|. \quad (26)$$

If $T_{j,0} = \emptyset$, the lower bound is trivially true. Combining Eq. (25) and (26), for $Q_j = \widehat{Q}_j$,

$$\sum_{t \in T_j} V_i\left(\left(\frac{w_{Q_j}}{1+\gamma}, Q_j\right), \beta_{-i}^t\right) \geq V_{j,0}|T_{j,0}| + W_{\widehat{Q}_j}|T_{j,1}|. \quad (27)$$

Case II: $Q_j = Q_j^*$. So, for any $t \in T_{j,1}$, using Lemma 9 (1) with $q = Q_j^*$, we get that $\left(\frac{w_{Q_j^*}}{1+\gamma}, Q_j^*\right)$ is allocated exactly Q_j^* units, which is the same as the allocation for $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$. Summing over all rounds in $T_{j,1}$,

$$\sum_{t \in T_{j,1}} V_i\left(\left(\frac{w_{Q_j}}{1+\gamma}, Q_j\right), \beta_{-i}^t\right) = W_{Q_j^*}|T_{j,1}|. \quad (28)$$

For the rounds in $T_{j,0}$, trivially,

$$\sum_{t \in T_{j,0}} V_i\left(\left(\frac{w_{Q_j}}{1+\gamma}, Q_j\right), \beta_{-i}^t\right) \geq 0. \quad (29)$$

Combining Eq. (28) and (29), for $Q_j = Q_j^*$,

$$\sum_{t \in T_j} V_i\left(\left(\frac{w_{Q_j}}{1+\gamma}, Q_j\right), \beta_{-i}^t\right) \geq W_{Q_j^*}|T_{j,1}|. \quad (30)$$

A.6.1 Proof of Lemma 9

(1) First we show that for any $t \in [T]$ and $q \leq r_t^*$, the 1-uniform bid $\left(\frac{w_q}{1+\gamma}, q\right)$ is allocated *exactly* q units. As $\left(\frac{w_q}{1+\gamma}, q\right) \in \mathcal{P}_1^*$, it is universally feasible. By assumption, $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ is allocated r_t^* units in round t . Let $\beta^t = (\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i}), \beta_{-i}^t)$. Recall that, $\beta_{-i,t}^{-(j)}$ is the j^{th} smallest winning bid in the absence of bids from bidder i for round t . If $r_t^* = 0$, the result is vacuously true. Suppose $r_t^* > 0$, then

$$\beta_{-i,t}^{-(q)} \leq \beta_{-i,t}^{-(r_t^*)} \leq p(\beta^t) \leq \frac{w_{r_t^*}}{1+\gamma} \leq \frac{w_q}{1+\gamma},$$

where the first inequality holds by definition of $\beta_{-i,t}^{-(j)}$ and our assumption that $q \leq r_t^*$. For the second inequality, let $Q_{\ell-1}^* < r_t^* \leq Q_\ell^*$. By Lemma 6,

1. If $p(\beta^t) = b_\ell^*$, we have $p(\beta^t) \geq \beta_{-i,t}^{-(r_t^*)}$ as b_ℓ^* is allocated r_t^* units.
2. If $p(\beta^t) = \beta_{-i,t}^{-(Q_\ell^*+1)}$, we have $r_t^* = Q_\ell^*$ and by definition of $\beta_{-i,t}^{-(j)}$, $p(\beta^t) = \beta_{-i,t}^{-(Q_\ell^*+1)} \geq \beta_{-i,t}^{-(r_t^*)}$.

The third is true as RoI constraints are satisfied by $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ for round t , and fourth is true as w_j is a non-decreasing function of j and $q \leq r_t^*$. From the first and last expressions, $\beta_{-i,t}^{-(q)} \leq \frac{w_q}{1+\gamma}$ which implies that $\left(\frac{w_q}{1+\gamma}, q\right)$ is allocated at least q units in round t . Moreover, $\left(\frac{w_q}{1+\gamma}, q\right)$ can be allocated at most q units. Combining both, gives the desired result that the 1-uniform bid $\left(\frac{w_q}{1+\gamma}, q\right)$ is allocated *exactly* q units.

(2) For any $j \in [m]$, let $T_j \subseteq [T]$, defined in Eq. (7), be the rounds in which b_j^* is the least winning bid. Suppose that $\exists t \in T_j$ such that $r_t^* < Q_j^*$. Then, we will show that in any $t' \in T_j$ in which $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ is allocated less than r_t^* units, i.e., $r_{t'}^* \leq r_t^*$, the 1-uniform bidding policy $\left(\frac{w_{r_t^*}}{1+\gamma}, r_t^*\right)$ is allocated *at least* $r_{t'}^*$ units.

Observe that $\left(\frac{w_{r_t^*}}{1+\gamma}, r_t^*\right) \in \mathcal{P}_1^*$, so it is UF. If $r_t^* = 0$, the result is trivially true. Fix any $j \in [m]$ and consider the set of rounds in T_j . As $r_t^* < Q_j^*$, by Lemma 6, $p(\beta^t) = b_j^*$. So, for any $t' \in T_j$ such that $r_{t'}^* \leq r_t^*$,

$$\beta_{-i,t'}^{-(r_{t'}^*)} \leq b_j^* \leq \frac{w_{r_t^*}}{1+\gamma}.$$

Here, the first inequality holds as b_j^* is the least winning bid for round t' and the second one holds as RoI constraints are valid for $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ for round t . From the first and the last expressions, we conclude that $\left(\frac{w_{r_t^*}}{1+\gamma}, r_t^*\right)$ is allocated *at least* $r_{t'}^*$ units.

A.7 Proof of Lemma 5

From Eq. (12) of the main text, for any $\mathbf{b} \in \mathcal{P}_m^*(\mathcal{B}_{-i})$, we have:

$$\sum_{t=1}^T V_i(\mathbf{b}, \beta_{-i}^t) \geq \sum_{j \in N^*} \sum_{t \in T_j} V_i\left(\left(\frac{w_{Q_j}}{1+\gamma}, Q_j\right), \beta_{-i}^t\right) + \sum_{j \in \widehat{N}} \sum_{t \in T_j} V_i\left(\left(\frac{w_{Q_j}}{1+\gamma}, Q_j\right), \beta_{-i}^t\right).$$

Substituting the lower bounds from Lemma 4,

$$\sum_{t=1}^T V_i(\mathbf{b}, \beta_{-i}^t) \geq \sum_{j \in N^*} \left(W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}} \left(V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right). \quad (31)$$

which gives:

$$\begin{aligned} \text{PoUF}_m(\mathcal{B}_{-i}) &\leq \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{\max_{\mathbf{b} \in \mathcal{P}_m^*(\mathcal{B}_{-i})} \sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t)} \\ &\stackrel{(31)}{\leq} \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{\sum_{j \in N^*} \left(W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}} \left(V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right)}. \end{aligned} \quad (32)$$

Let N_0^*, \widehat{N}_0 be the partition of $[m]$ that minimizes the RHS of (32). Then,

$$V_m^{\text{OPT}}(\mathcal{B}_{-i}) = \sum_{t=1}^T \sum_{\ell=0}^1 V_{j,\ell} |T_{j,\ell}| = \sum_{j \in N_0^*} \left(V_{j,0} |T_{j,0}| + W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}_0} \left(V_{j,0} |T_{j,0}| + W_{Q_j^*} |T_{j,1}| \right). \quad (33)$$

Consider a partition of $[m]$ that differs from the maximizing partition (N_0^*, \widehat{N}_0) by exactly one element, i.e., for some $a \in \widehat{N}_0$, consider the following partition: $(N_0^* \cup \{a\}, \widehat{N}_0 \setminus \{a\})$. By definition,

$$\begin{aligned} \sum_{j \in N_0^*} \left(W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}_0} \left(V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right) &\geq \sum_{j \in N_0^* \cup \{a\}} \left(W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}_0 \setminus \{a\}} \left(V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right) \\ &\implies V_{a,0} |T_{a,0}| + W_{\widehat{Q}_a} |T_{a,1}| \geq W_{Q_a^*} |T_{a,1}|. \end{aligned} \quad (34)$$

Now, for some $b \in N_0^*$, consider the following partition: $(N_0^* \setminus \{b\}, \widehat{N}_0 \cup \{b\})$. By definition,

$$\begin{aligned} \sum_{j \in N_0^*} \left(W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}_0} \left(V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right) &\geq \sum_{j \in N_0^* \setminus \{b\}} \left(W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}_0 \cup \{b\}} \left(V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right) \\ &\implies W_{Q_b^*} |T_{b,1}| - W_{\widehat{Q}_b} |T_{b,1}| \geq V_{b,0} |T_{b,0}|. \end{aligned} \quad (35)$$

Plugging in the values,

$$\begin{aligned} \text{PoUF}_m(\mathcal{B}_{-i}) &\stackrel{(33)}{\leq} \frac{\sum_{j \in N_0^*} \left(V_{j,0} |T_{j,0}| + W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}_0} \left(V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right)}{\sum_{j \in N_0^*} \left(W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}_0} \left(V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right)} \\ &\stackrel{(34)}{\leq} \frac{\sum_{j \in N_0^*} \left(V_{j,0} |T_{j,0}| + W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}_0} \left(2V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right)}{\sum_{j \in N_0^*} \left(W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}_0} \left(V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right)} \\ &\stackrel{(35)}{\leq} \frac{\sum_{j \in N_0^*} \left(2W_{Q_j^*} |T_{j,1}| - W_{\widehat{Q}_j} |T_{j,1}| \right) + \sum_{j \in \widehat{N}_0} \left(2V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right)}{\sum_{j \in N_0^*} \left(W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}_0} \left(V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right)} \\ &= \frac{2 \left\{ \sum_{j \in N_0^*} W_{Q_j^*} |T_{j,1}| + \sum_{j \in \widehat{N}_0} \left(V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right) \right\} - \sum_{j=1}^m W_{\widehat{Q}_j} |T_{j,1}|}{\sum_{j \in N_0^*} \left(W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}_0} \left(V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right)} \\ &= 2 - \theta_{\mathcal{B}_{-i}}, \end{aligned}$$

where

$$\begin{aligned} 0 < \theta_{\mathcal{B}_{-i}} &= \frac{\sum_{j=1}^m W_{\widehat{Q}_j} |T_{j,1}|}{\sum_{j \in N_0^*} \left(W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}_0} \left(V_{j,0} |T_{j,0}| + W_{\widehat{Q}_j} |T_{j,1}| \right)} \\ &\stackrel{(34)}{\leq} \frac{\sum_{j=1}^m W_{\widehat{Q}_j} |T_{j,1}|}{\sum_{j \in N_0^*} \left(W_{Q_j^*} |T_{j,1}| \right) + \sum_{j \in \widehat{N}_0} \left(W_{Q_j^*} |T_{j,1}| \right)} \\ &= \frac{\sum_{j=1}^m W_{\widehat{Q}_j} |T_{j,1}|}{\sum_{j=1}^m W_{Q_j^*} |T_{j,1}|} \\ &\stackrel{(a)}{\leq} \max_{j \in [m]} \frac{W_{\widehat{Q}_j}}{W_{Q_j^*}} \leq 1, \end{aligned}$$

and (a) follows from Fact 1.

Fact 1 (Williamson and Shmoys (2011, pp. 25)). *For positive numbers a_1, \dots, a_k and b_1, \dots, b_k ,*

$$\frac{\sum_{j=1}^k a_j}{\sum_{j=1}^k b_j} \leq \max_{j \in [k]} \frac{a_j}{b_j}.$$

Finally,

$$\text{POUF}_m = \max_{\mathcal{B}_{-i}} \text{POUF}_m(\mathcal{B}_{-i}) \leq 2 - \min_{\mathcal{B}_{-i}} \theta_{\mathcal{B}_{-i}} =: 2 - \theta,$$

where $\theta = \min_{\mathcal{B}_{-i}} \theta_{\mathcal{B}_{-i}} \in (0, 1]$.

A.8 Proof of Theorem 6.1

We define the following metrics for any \mathcal{B}_{-i} :

$$\text{RATIO-UNI}_m(\mathcal{B}_{-i}) := \frac{V_m^*(\mathcal{B}_{-i})}{V_1^*(\mathcal{B}_{-i})}, \quad \text{and} \quad \text{RATIO-OPT}_m(\mathcal{B}_{-i}) := \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_1^{\text{OPT}}(\mathcal{B}_{-i})}.$$

So, $\text{RATIO-UNI}_m = \max_{\mathcal{B}_{-i}} \text{RATIO-UNI}_m(\mathcal{B}_{-i})$ and $\text{RATIO-OPT}_m = \max_{\mathcal{B}_{-i}} \text{RATIO-OPT}_m(\mathcal{B}_{-i})$. We first derive bounds on $\text{RATIO-UNI}_m(\mathcal{B}_{-i})$ and $\text{RATIO-OPT}_m(\mathcal{B}_{-i})$ and then maximizing over all \mathcal{B}_{-i} , we get the desired result. Note that the lower bound for $\text{RATIO-UNI}_m(\mathcal{B}_{-i})$ holds true as $\mathcal{P}_1^* \subseteq \mathcal{P}_m^*$ and the lower bound for $\text{RATIO-OPT}_m(\mathcal{B}_{-i})$ holds true by definition of the Problem (VM-OPT_m).

Claim 3 (Upper bound on $\text{RATIO-UNI}_m(\mathcal{B}_{-i})$ and $\text{RATIO-OPT}_m(\mathcal{B}_{-i})$). *For any fixed \mathcal{B}_{-i} and $m \geq 2$, $\text{RATIO-UNI}_m(\mathcal{B}_{-i}) < m$ and $\text{RATIO-OPT}_m(\mathcal{B}_{-i}) < m$.*

A.8.1 Bound for $\text{RATIO-OPT}_m(\mathcal{B}_{-i})$.

Fix some bid history $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$. To show the upper bound, without loss of generality, assume $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i}) = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$. So, by invoking Lemma 8 (as $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ is feasible),

$$V_m^{\text{OPT}}(\mathcal{B}_{-i}) = \sum_{t=1}^T V_i(\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i}); \beta_{-i}^t) = \sum_{t=1}^T \max_{j \in [m]} V_i((b_j, Q_j); \beta_{-i}^t).$$

Fact 2. *Let $a_1, \dots, a_m \geq 0$, then $\max_{j \in [m]} a_j < \sum_{j=1}^m a_j$ unless $\exists k \in [m]$ such that $a_j = 0, \forall j \neq k$.*

Consider the two following cases:

Case 1. Suppose, $\exists t \in [T]$ such that $\max_{j \in [m]} V_i((b_j, Q_j); \beta_{-i}^t) < \sum_{j=1}^m V_i((b_j, Q_j); \beta_{-i}^t)$. Then,

$$\begin{aligned} V_m^{\text{OPT}}(\mathcal{B}_{-i}) &< \sum_{t=1}^T \sum_{j=1}^m V_i((b_j, Q_j); \beta_{-i}^t) \\ &= \sum_{j=1}^m \sum_{t=1}^T V_i((b_j, Q_j); \beta_{-i}^t) \\ &\leq \sum_{j=1}^m \max_{\mathbf{b}} \sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t) = m V_1^{\text{OPT}}(\mathcal{B}_{-i}) \end{aligned} \quad (36)$$

where in Eq. (36), the maximum is taken over all 1-uniform feasible bidding policies for \mathcal{B}_{-i} .

Case 2. Suppose for all $t \in [T]$, $\max_{j \in [m]} V_i((b_j, Q_j); \beta_{-i}^t) = \sum_{j=1}^m V_i((b_j, Q_j); \beta_{-i}^t)$ and define $k = \operatorname{argmax}_{j \in [m]} V_i((b_j, Q_j); \beta_{-i}^t)$. So,

$$V_m^{\text{OPT}}(\mathcal{B}_{-i}) = \sum_{t=1}^T V_i((b_k, Q_k); \beta_{-i}^t) \leq \max_{\mathbf{b}} \sum_{t=1}^T V_i(\mathbf{b}; \beta_{-i}^t) = V_1^{\text{OPT}}(\mathcal{B}_{-i}). \quad (37)$$

In Eq. (37), the maximum is taken over all 1-uniform feasible bidding policies for \mathcal{B}_{-i} . By Problem (VM-OPT $_m$), $V_m^{\text{OPT}}(\mathcal{B}_{-i}) \geq V_1^{\text{OPT}}(\mathcal{B}_{-i})$. So, in **Case 2**, $V_m^{\text{OPT}}(\mathcal{B}_{-i}) = V_1^{\text{OPT}}(\mathcal{B}_{-i})$. From the two cases, we conclude that $\frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_1^{\text{OPT}}(\mathcal{B}_{-i})} < m$ for any \mathcal{B}_{-i} . Maximizing over all \mathcal{B}_{-i} , we get the desired result.

A.8.2 Bounds on RATIO-UNI $_m(\mathcal{B}_{-i})$.

Fix a bid history $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$. To show the upper bound, without loss of generality, assume $\mathbf{b}_m^*(\mathcal{B}_{-i}) = \left\langle \left(\frac{w_{Q_1}}{1+\gamma}, q_1 \right), \dots, \left(\frac{w_{Q_m}}{1+\gamma}, q_m \right) \right\rangle$. So, by invoking Lemma 3,

$$V_m^*(\mathcal{B}_{-i}) = \sum_{t=1}^T V_i(\mathbf{b}_m^*(\mathcal{B}_{-i}), \beta_{-i}^t) = \sum_{t=1}^T \max_{j \in [m]} V_i\left(\left(\frac{w_{Q_j}}{1+\gamma}, Q_j\right); \beta_{-i}^t\right).$$

Henceforth, the proof proceeds identical to that for establishing the upper bound for RATIO-OPT $_m(\mathcal{B}_{-i})$, with the exception that in Eq. (36) and Eq. (37), the maximum is taken over the UF bidding policies in \mathcal{P}_1^* , instead of all feasible bidding policies. Thus, we conclude that $\frac{V_m^*(\mathcal{B}_{-i})}{V_1^*(\mathcal{B}_{-i})} < m$. Maximizing over all \mathcal{B}_{-i} , we get the desired result.

A.9 Proof of Corollary 7.1

Let $\mathbf{b} = \left\langle \left(\frac{w_{Q_1}}{1+\gamma}, q_1 \right), \dots, \left(\frac{w_{Q_k}}{1+\gamma}, q_k \right) \right\rangle$ be a nested UF policy in \mathcal{P}_m^* where $k \in [m]$. Hence, for any $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$, by Lemma 3, we have

$$\begin{aligned} \max_{\mathbf{b} \in \mathcal{P}_m^*} \sum_{t=1}^T V_i(\mathbf{b}, \beta_{-i}^t) &= \max_{\mathbf{b} \in \mathcal{P}_m^*} \sum_{t=1}^T V_i\left(\left\langle \left(\frac{w_{Q_1}}{1+\gamma}, q_1 \right), \dots, \left(\frac{w_{Q_k}}{1+\gamma}, q_k \right) \right\rangle, \beta_{-i}^t\right) \\ &= \max_{\mathbf{b} \in \mathcal{P}_m^*} \sum_{t=1}^T \max_{\ell \in [k]} V_i\left(\left(\frac{w_{Q_\ell}}{1+\gamma}, Q_\ell\right), \beta_{-i}^t\right) \\ &= \max_{S \subseteq [M]: |S| \leq m} \sum_{t=1}^T \max_{\ell \in S} V_i\left(\left(\frac{w_\ell}{1+\gamma}, \ell\right), \beta_{-i}^t\right). \end{aligned}$$

B Tight Lower Bounds

B.1 Tight Lower Bound for Theorem 5.2 (For $m = 1$)

In this section, we construct a bid history, \mathcal{B}_{-i} and valuation curve \mathbf{v} (equivalently the \mathcal{P}_m^*) for which the upper bound on POUF $_m$, presented in Theorem 5.2, is tight. Recall that for any choice of \mathcal{B}_{-i} , $\text{POUF}_m(\mathcal{B}_{-i}) \leq 2 - \theta_{\mathcal{B}_{-i}}$, where $\theta_{\mathcal{B}_{-i}} \leq \max_{j \in [m]} (W_{\hat{Q}_j} / W_{Q_j^*})$. To minimize $\theta_{\mathcal{B}_{-i}}$, we choose a valuation curve that is very weakly decreasing. We then set the competing bids such that $\hat{Q}_j \ll Q_j^*$ for all $j \in [m]$. Finally, we determine the values of $|T_{j,0}|$ and $|T_{j,1}|$ for which the upper bound is tight. See the definition of these quantities in Section 5.1, where the first part of the theorem is proven.

We present the case when $m = 1$ below. The case for $m \geq 2$ is deferred to Appendix B.2. Although the main idea for both the cases are the same, for $m \geq 2$, the construction is more involved, and hence presented separately (see details in Appendix B.2). Formally, for any $\delta \in (0, 1/2]$, we design a bid history, \mathcal{B}_{-i} , and valuation vector, \mathbf{v} for which $\text{POUF}_1 \geq 2 - \delta$.

Let $\bar{M} = 2 \lceil 1/\delta \rceil$. Suppose $\mathbf{v} = [1, v, \dots, v] \in \mathbb{R}^{\bar{M}}$, target RoI $\gamma = 0$ and $v = 1 - 4\epsilon$ where $\epsilon = \frac{\delta - 1/\bar{M}}{4(1 - 1/\bar{M})} < \frac{\delta}{4}$. Observe that $\epsilon \in (0, 1/8)$ as $\delta \leq 1/2$. Let $T = \bar{M}$ and $K = \bar{M} + 1$. The bid history is defined as:

$$\beta_{-i,t}^{-(j)} = \begin{cases} 1 - \epsilon, & \text{if } t \leq \bar{M} - 1 \text{ and } j = 1 \\ C, & \text{if } t \leq \bar{M} - 1 \text{ and } 2 \leq j \leq K, \\ \epsilon, & \text{if } t = \bar{M} \text{ and } 1 \leq j \leq K \end{cases}$$

where $C \gg w_1$. The bid history is presented in Table 2.

Table 2: Bid history, \mathcal{B}_{-i} , that achieves the 2-approximation asymptotically for $m = 1$. Here $C \gg w_1$ and $\epsilon > 0$ is a small real number.

$t = 1$	$t = 2$	\dots	$t = \bar{M} - 1$	$t = \bar{M}$
C	C	\dots	C	ϵ
\vdots	\vdots	\dots	\vdots	\vdots
C	C	\dots	C	\vdots
$1 - \epsilon$	$1 - \epsilon$	\dots	$1 - \epsilon$	ϵ

Notice that the constructed \mathcal{B}_{-i} contains $\bar{M} - 1$ rounds with high competition (where the bidder can acquire at most 1 unit), while there is one round with minimal competition, allowing the bidder to obtain any desired number of units. In such instances, 1-uniform UF bidding policies of the form (w_q, q) do not perform well universally on all rounds. This is because under the 1-uniform UF bidding policies of the form (w_q, q) , as q increases, despite the increasing demand (i.e., q), the bid value w_q decreases, thereby reducing the likelihood of acquiring a higher number of units.

Computing $V_1^{\text{OPT}}(\mathcal{B}_{-i})$. Note that no feasible bidding policy can be allocated more than 1 unit in the first $\bar{M} - 1$ rounds and \bar{M} units in the final round. We claim that $\mathbf{b}_1^{\text{OPT}}(\mathcal{B}_{-i}) = (1, \bar{M})$. Observe that $\mathbf{b}_1^{\text{OPT}}(\mathcal{B}_{-i})$ is allocated 1 unit in each of the first $\bar{M} - 1$ rounds and \bar{M} units in the final round. Hence, it is allocated the maximum number of units possible. To verify that $\mathbf{b}_1^{\text{OPT}}(\mathcal{B}_{-i})$ is feasible, notice that for $\forall t \in [\bar{M} - 1]$, $p(\beta^t) = 1 \leq w_1 = 1$. For round $t = \bar{M}$, as $K > \bar{M}$, $p(\beta^t) = \epsilon \leq 1 - 4\epsilon < w_{\bar{M}} = \lambda \cdot 1 + (1 - \lambda) \cdot (1 - 4\epsilon)$ where $\lambda = 1/\bar{M}$, implying that $\mathbf{b}_1^{\text{OPT}}(\mathcal{B}_{-i})$ is feasible. So, $V_1^{\text{OPT}}(\mathcal{B}_{-i}) = \bar{M} + v(\bar{M} - 1)$.

Computing the optimal policy in \mathcal{P}_1^* . Here, we argue that the optimal policy in \mathcal{P}_1^* is $(w_1, 1) = (1, 1)$. Observe that the policy $(1, 1)$ is allocated 1 unit in each round. Hence, it obtains a total value of \bar{M} .

As $\beta_{-i,t}^{-(1)} = 1 - \epsilon > 1 - 2\epsilon = w_2$, for $t = 1, 2, \dots, \bar{M} - 1$, bidding (w_q, q) for $q \geq 2$ does not get any value in the first $\bar{M} - 1$ rounds. Bidding (w_q, q) gets exactly q units in round \bar{M} as $w_q > \epsilon$ for any $q \geq 2$. So, for $2 \leq q \leq \bar{M}$, the total value obtained by bidding (w_q, q) is $1 + (q - 1)v < q$. Hence, $V_1^*(\mathcal{B}_{-i}) = \bar{M}$, which is the value obtained by the UF bid $(w_1, 1)$. So,

$$\text{PoUF}_1 \geq \frac{V_1^{\text{OPT}}(\mathcal{B}_{-i})}{V_1^*(\mathcal{B}_{-i})} = \frac{\bar{M} + v(\bar{M} - 1)}{\bar{M}} = \frac{2\bar{M} - 1 - 4\epsilon(\bar{M} - 1)}{\bar{M}} = 2 - \delta.$$

B.2 Tight Lower Bound for Theorem 5.2 (For $m \geq 2$)

In this section, we design a bid history \mathcal{B}_{-i} and valuation vector, \mathbf{v} such that for any $\delta \in (0, 1/2]$, $\text{PoUF}_m \geq 2 - \delta$. By definition, for any \mathcal{B}_{-i} and $m \geq 2$,

$$\text{PoUF}_m \geq \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_m^*(\mathcal{B}_{-i})} > \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{mV_1^*(\mathcal{B}_{-i})}, \quad (38)$$

where the second inequality follows directly from the bounds on $\text{RATIO-UNI}_m(\mathcal{B}_{-i})$ (see details in Appendix A.8). So, instead of computing $V_m^*(\mathcal{B}_{-i})$ directly which can be non-trivial, we obtain $V_1^*(\mathcal{B}_{-i})$ and show that the bound is tight.

B.2.1 Construction of \mathcal{B}_{-i} .

We first decide all the parameters.

- Fix $m \geq 2$ and any integer $N \geq 2 \lceil \frac{1}{\delta} \rceil$.
- Let $\bar{M} = N^{2m-1}$. Consider $T = N^{2m-1}$ rounds and $K = N^{2m-1} + 1$ units in each auction.
- Let $\epsilon' = \frac{m\delta/(2m-1)-1/N}{2(1-1/N)} < \frac{\delta}{2} \leq \frac{1}{4}$. Set ϵ such that $\epsilon' = \epsilon N^{2m-1}(N^{2m-1} + 1)$.

Consider a valuation vector $\mathbf{v} = [1, v, \dots, v]$ such that $v = 1 - 2\epsilon'$, and target RoI $\gamma = 0$. Partition the N^{2m-1} rounds into $2m$ partitions such that the first partition has 1 round and the j^{th} partition has $N^{j-1} - N^{j-2}$ rounds for $2 \leq j \leq 2m$. Each partition has identical competing bid profile submitted by other bidders. In particular,

1. The first partition (containing one round) has all the bids submitted by others as $w_{N^{2m-1}+1} + \epsilon$.
2. If $j > 1$ and j is odd, for the j^{th} partition (of size $N^{j-1} - N^{j-2}$), the smallest $N^{2m-j} + 1$ winning competing bids are $w_{N^{2m-j}+1} + \epsilon$ and the remaining bids are $C \gg w_1$.
3. If $j > 1$ and j is even, for the j^{th} partition (of size $N^{j-1} - N^{j-2}$), the smallest N^{2m-j} winning competing bids are $w_{N^{2m-j}+1} + \epsilon$ and the remaining bids are $C \gg w_1$.

We present an example for such a bid history in Table 3.

Table 3: Example of bid history, \mathcal{B}_{-i} achieving 2-approximation asymptotically for $m = 2$. Each round in the same partition has identical competing bid profile. Total number of units in each auction is $K = N^3 + 1$.

Partition 1 $t = 1$	Partition 2 $t \in [2, N]$	Partition 3 $t \in [N + 1, N^2]$	Partition 4 $t \in [N^2 + 1, N^3]$
0 bids are C	$N^3 - N^2 + 1$ bids are C	$N^3 - N$ bids are C	N^3 bids are C
$N^3 + 1$ bids are $w_{N^3+1} + \epsilon$	N^2 bids are $w_{N^2+1} + \epsilon$	$N + 1$ bids are $w_{N+1} + \epsilon$	1 bid is $w_2 + \epsilon$

B.2.2 Computing $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$.

We make the following claim about the optimal m -uniform bidding policy for the constructed \mathcal{B}_{-i} .

Lemma 10. For the aforementioned \mathcal{B}_{-i} , $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i}) = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$ where

$$(b_j, q_j) = \begin{cases} (1, N), & \text{if } j = 1 \\ (w_{N^{2j-2}}, N^{2j-1} - N^{2j-3}), & \text{if } 2 \leq j \leq m. \end{cases} \quad (39)$$

Furthermore,

$$V_m^{\text{OPT}}(\mathcal{B}_{-i}) = N^{2m-1} + (2m-1)(N^{2m-1} - N^{2m-2})v.$$

Proof. Proof. We begin by a crucial observation that the bid history does not allow obtaining more than N^{2m-j} units in the j^{th} partition *irrespective of the number of bids* submitted by the bidder. To verify this, note that, the maximum number of units that can be allocated to any bidding policy in the j^{th} partition is either N^{2m-j} or $N^{2m-j} + 1$ (depending on if j is even or odd). Suppose contrary to our claim, the bidder is allocated $N^{2m-j} + 1$ units in the some round t in the j^{th} partition by bidding some \mathbf{b} . Let $\beta^t = (\mathbf{b}, \beta_{-i}^t)$ be the complete bid profile. So, $p(\beta^t) \geq w_{N^{2m-j+1}} + \epsilon$ but $x_i(\beta^t) = N^{2m-j} + 1$ indicating that the RoI constraint is violated, which verifies our claim.

So, the total number of units, N_{total} , that can be obtained by the bidder across the auctions is:

$$N_{\text{total}} \leq N^{2m-1} + \sum_{j=2}^{2m} N^{2m-j}(N^{j-1} - N^{j-2}) = 2mN^{2m-1} - (2m-1)N^{2m-2} =: N_{\text{max}}.$$

Now, we compute the the number of units obtained by bidding $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ and show that it is allocated N_{max} units across all the auctions, demonstrating that it is the optimal bidding strategy.

Consider any auction in the j^{th} partition. The lowest winning bid in the bid profile is $w_{N^{2m-j+1}} + \epsilon$. Note that the unique bid values (ignoring the quantity for the sake of brevity) in $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ are $\mathbf{b} = \{1, w_{N^2}, \dots, w_{N^{2m-2}}\}$. We claim that the winning bid values of $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ in the j^{th} partition are $\hat{\mathbf{b}} = \{1, w_{N^2}, \dots, w_{N^{2m+2\lfloor -j/2 \rfloor}}\}$. This is true because the least bid value in $\hat{\mathbf{b}}$ is greater than $w_{N^{m-j+1}} + \epsilon$, i.e.,

$$\begin{aligned} w_{N^{2m+2\lfloor -j/2 \rfloor}} - (w_{N^{2m-j+1}} + \epsilon) &\geq w_{N^{2m-j}} - (w_{N^{2m-j+1}} + \epsilon) \\ &= \frac{1-v}{N^{2m-j}(N^{2m-j}+1)} - \epsilon = \frac{2\epsilon N^{2m-1}(N^{2m-1}+1)}{N^{2m-j}(N^{2m-j}+1)} - \epsilon \geq \epsilon > 0. \end{aligned}$$

Let N_j denote the number of units allocated to $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ in each auction in the j^{th} partition. There are two cases:

- (a) for j odd, recall that for the j^{th} partition (of size $N^{j-1} - N^{j-2}$), the smallest $N^{2m-j} + 1$ winning competing bids are $w_{N^{2m-j+1}} + \epsilon$ and the remaining bids are $C \gg w_1$. Then,

$$N_j = N + \sum_{\ell=2}^{m-\frac{j-1}{2}} (N^{2\ell-1} - N^{2\ell-3}) = N^{2m-j}.$$

- (b) For j even, recall that for the j^{th} partition (of size $N^{j-1} - N^{j-2}$), the smallest N^{2m-j} winning competing bids are $w_{N^{2m-j+1}} + \epsilon$ and the remaining bids are $C \gg w_1$.

$$N_j = \min \left\{ N^{2m-j}, N + \sum_{\ell=2}^{m+\frac{1-j}{2}} (N^{2\ell-1} - N^{2\ell-3}) \right\} = \min\{N^{2m-j}, N^{2m-j+1}\} = N^{2m-j}.$$

Here, the minimum is taken over two quantities as the first quantity is the number of finite competing bids in any round t in the j^{th} partition and the second quantity represents the total demand of the winning bids in $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$. So, the total number of units obtained across all rounds:

$$N(\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})) = N^{2m-1} + \sum_{j=2}^{2m} N^{2m-j}(N^{j-1} - N^{j-2}) = 2mN^{2m-1} - (2m-1)N^{2m-2}.$$

As this is the maximum number of units that can be obtained by the bidder, $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ is optimal. The total value obtained by bidding $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ is

$$\begin{aligned} V_m^{\text{OPT}}(\mathcal{B}_{-i}) &= 1 + (N^{2m-1} - 1)v + \sum_{j=2}^{2m} (N^{j-1} - N^{j-2})(1 + (N^{2m-j} - 1)v) \\ &= N^{2m-1} + (2m-1)(N^{2m-1} - N^{2m-2})v. \end{aligned}$$

□

B.2.3 Computing $V_1^*(\mathcal{B}_{-i})$

Recall that we chose to compute $V_1^*(\mathcal{B}_{-i})$ and then invoke the bounds on $\text{RATIO-UNI}_m(\mathcal{B}_{-i})$, instead of directly evaluating $V_m^*(\mathcal{B}_{-i})$.

Lemma 11. *For the aforementioned \mathcal{B}_{-i} , $\mathbf{b}_1^*(\mathcal{B}_{-i}) = (1, 1)$ and $V_1^*(\mathcal{B}_{-i}) = N^{2m-1}$.*

Proof. Proof. The basic idea is to enumerate the total units (value) that can be obtained by bidding (w_q, q) for $q \in [N^{2m-1}]$ and then finding the maximum of those values. As q can be exponential in m , we exploit the structure of the bid history to compute the objective in an efficient manner.

Suppose $q = 1$. The maximum number of units (value) that can be obtained by bidding $(1, 1)$ is trivially N^{2m-1} . So, $(1, 1)$ obtains a total value N^{2m-1} .

Suppose $q \geq 2$. Furthermore, assume $N^{2m-j} < q \leq N^{2m-j+1}$, for some $2 \leq j \leq 2m$. Consider any bid of the form (w_q, q) . Bidding (w_q, q) does not obtain any units in the partitions indexed by $j, j+1, \dots, 2m$ as w_q is strictly less than the least winning competing bids in those partitions, i.e., $w_q \leq w_{N^{2m-j+1}} < w_{N^{2m-\ell+1}} + \epsilon$, for any $\ell \in \{j, j+1, \dots, 2m\}$. So, if $N^{2m-j} < q \leq N^{2m-j+1}$, (w_q, q) gets no units in

$$\sum_{\ell=j}^{2m} N^{\ell-1} - N^{\ell-2} = N^{2m-1} - N^{j-2} \text{ auctions.}$$

In the remaining N^{j-2} auctions it can win at most q units. So, the maximum value obtained by (w_q, q) for any given $q \geq 2$ is

$$N^{j-2}(1 + (q-1)v) \leq N^{j-2}(1 + (N^{2m-j+1} - 1)v) = N^{j-2} + (N^{2m-1} - N^{j-2})v < N^{2m-1},$$

where the last inequality holds as $v < 1$. Hence, $\mathbf{b}_1^*(\mathcal{B}_{-i}) = (1, 1)$ and $V_1^*(\mathcal{B}_{-i}) = N^{2m-1}$. □

Hence, from Lemma 10, Lemma 11 and Eq. (38)

$$\begin{aligned} \text{POUF}_m &> \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{mV_1^*(\mathcal{B}_{-i})} = \frac{N^{2m-1} + (2m-1)(N^{2m-1} - N^{2m-2})v}{mN^{2m-1}} \\ &= \frac{2mN^{2m-1} - (2m-1)N^{2m-2} - 2\epsilon'(2m-1)(N^{2m-1} - N^{2m-2})}{mN^{2m-1}} \end{aligned}$$

$$\begin{aligned}
&= 2 - \frac{2m-1}{m} \left(\frac{1}{N} + 2\epsilon' \left(1 - \frac{1}{N} \right) \right) \\
&= 2 - \delta,
\end{aligned}$$

where the last step follows by substituting the value of ϵ' and thus concludes the proof.

B.3 Tight Lower Bound for Theorem 6.1

In this section, we construct a bid history \mathcal{B}_{-i} and valuation vector, \mathbf{v} such that for any $\delta \in (0, 1/2]$, $\text{RATIO-OPT}_m \geq m - \delta$ and $\text{RATIO-UNI}_m \geq m - \delta$. The same example serves both the purposes.

B.3.1 Construction of \mathcal{B}_{-i} .

We first decide all the parameters.

- Fix $m \geq 2$ and any integer $N \geq \lceil \frac{m}{\delta} \rceil$.
- Let $\bar{M} = N^{m-1}$. Consider $T = N^{m-1}$ rounds and $K = N^{m-1}$ units in each auction.
- Let $\epsilon' = \frac{\delta/(m-1)-1/N}{2(1-1/N)} < \frac{\delta}{2(m-1)} \leq \frac{1}{4}$. Set ϵ such that $\epsilon' = \epsilon N^{m-1}(N^{m-1} + 1)$.

Let the valuation vector be $\mathbf{v} = [1, v, \dots, v]$ such that $v = 1 - 2\epsilon'$, and target RoI, $\gamma = 0$. Partition the N^{m-1} rounds into m partitions such that the first partition has 1 round and the j^{th} partition has $N^{j-1} - N^{j-2}$ rounds for $2 \leq j \leq m$. Each partition has identical competing bid profile submitted by the other bidders. In particular,

1. The first partition (containing one round) has all the competing bids to be $w_{N^{m-1}+1} + \epsilon$.
2. For $2 \leq j \leq m$, the j^{th} partition (of size $N^{j-1} - N^{j-2}$), the smallest $N^{m-j} + 1$ competing winning bids are $w_{N^{m-j}+1} + \epsilon$ and the remaining bids are $C \gg w_1$.

We present an example for such a bid history in Table 4.

Table 4: Example of bid history, \mathcal{B}_{-i} achieving 4-approximation asymptotically for $m = 4$. Each round in the same partition has identical competing bid profile. Total number of units in each auction is $K = N^3$.

Partition 1 $t = 1$	Partition 2 $t \in [2, N]$	Partition 3 $t \in [N + 1, N^2]$	Partition 4 $t \in [N^2 + 1, N^3]$
0 bids are C N^3 bids are $w_{N^3+1} + \epsilon$	$N^3 - N^2 - 1$ bids are C $N^2 + 1$ bids are $w_{N^2+1} + \epsilon$	$N^3 - N - 1$ bids are C $N + 1$ bids are $w_{N+1} + \epsilon$	$N^3 - 2$ bids are C 2 bids are $w_2 + \epsilon$

B.3.2 Identifying $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ (equivalently $\mathbf{b}_m^*(\mathcal{B}_{-i})$)

We make the following claim about the optimal m -uniform (UF) bidding policy for \mathcal{B}_{-i} .

Lemma 12. For the aforementioned \mathcal{B}_{-i} , $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i}) = \mathbf{b}_m^*(\mathcal{B}_{-i}) = \langle (b_1, q_1), \dots, (b_m, q_m) \rangle$ where

$$(b_j, q_j) = \begin{cases} (1, 1), & \text{if } j = 1 \\ (w_{N^{j-1}}, N^{j-1} - N^{j-2}), & \text{if } 2 \leq j \leq m \end{cases}. \quad (40)$$

Furthermore,

$$V_m^{\text{OPT}}(\mathcal{B}_{-i}) = V_m^*(\mathcal{B}_{-i}) = N^{m-1} + (m-1)(N^{m-1} - N^{m-2})v.$$

Proof. Proof. We begin by a crucial observation that the bid history does not allow obtaining more than N^{m-j} units in the j^{th} partition *irrespective of the number of bids* submitted by the bidder. To verify this, suppose contrary to our claim, that the bidder is allocated $N^{m-j} + 1$ units in the some round t in the j^{th} partition by bidding some \mathbf{b} . Let $\beta^t = (\mathbf{b}, \beta_{-i}^t)$ be the complete bid profile. Hence, $p(\beta^t) \geq w_{N^{m-j+1}} + \epsilon$ but $x_i(\beta^t) = N^{m-j} + 1$ indicating that the RoI constraint is violated which verifies our claim.

So, the total number of units, N_{total} , that can be obtained by the bidder across the auctions is:

$$N_{\text{total}} \leq N^{m-1} + \sum_{j=2}^m N^{m-j}(N^{j-1} - N^{j-2}) = mN^{m-1} - (m-1)N^{m-2} := N_{\text{max}}.$$

Now, we compute the number of units obtained by bidding $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ and show that it is allocated N_{max} units across all the auctions, showing that it is the optimal bidding strategy. Moreover, $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ also exhibits a nested structure indicating that it is also UF which implies that $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i}) = \mathbf{b}_m^*(\mathcal{B}_{-i})$. Hence, the feasibility of $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ is satisfied.

To show $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ is optimal, consider any auction in the j^{th} partition. The lowest winning competing bid is $w_{N^{m-j+1}} + \epsilon$. Note that the unique bid values (ignoring the quantity for the sake of brevity) in $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$, provided in Eq. (40), are $\mathbf{b} = \{1, w_N, \dots, w_{N^{m-1}}\}$. We claim that the winning bid values of $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ in the j^{th} partition are $\hat{\mathbf{b}} = \{1, w_N, \dots, w_{N^{m-j}}\}$. Again recall that for $2 \leq j \leq m$, for the j^{th} partition (of size $N^{j-1} - N^{j-2}$), the smallest $N^{m-j} + 1$ competing winning bids are $w_{N^{m-j+1}} + \epsilon$ and the remaining bids are $C \gg w_1$. And here, the least bid value in $\hat{\mathbf{b}}$ is greater than $w_{N^{m-j+1}} + \epsilon$, i.e.,

$$w_{N^{m-j}} - (w_{N^{m-j+1}} + \epsilon) = \frac{1-v}{N^{m-j}(N^{m-j}+1)} - \epsilon = \frac{2\epsilon N^{m-1}(N^{m-1}+1)}{N^{m-j}(N^{m-j}+1)} - \epsilon \geq \epsilon > 0.$$

Moreover, observe that the bidder is allocated the maximum number of units demanded for each of the bid value in $\hat{\mathbf{b}}$. So, the number of units in each auction in the j^{th} partition by bidding $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ is

$$N_j = 1 + \sum_{\ell=2}^{m-j+1} (N^{\ell-1} - N^{\ell-2}) = N^{m-j}.$$

Hence, the total number of units obtained across all rounds:

$$N(\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})) = N^{m-1} + \sum_{j=2}^m N^{m-j}(N^{j-1} - N^{j-2}) = mN^{m-1} - (m-1)N^{m-2} = N_{\text{max}}.$$

As this is the maximum number of units that can be obtained by the bidder, $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ is optimal. The total value obtained by bidding $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ is given by

$$\begin{aligned} V_m^{\text{OPT}}(\mathcal{B}_{-i}) &= 1 + (N^{m-1} - 1)v + \sum_{j=2}^m (N^{j-1} - N^{j-2})(1 + (N^{m-j} - 1)v) \\ &= N^{m-1} + (m-1)(N^{m-1} - N^{m-2})v. \end{aligned}$$

□

B.3.3 Computing $\mathbf{b}_1^{\text{OPT}}(\mathcal{B}_{-i})$ (equivalently $\mathbf{b}_1^*(\mathcal{B}_{-i})$).

In this section, we identify the optimal 1-uniform bidding (UF) policy for the bid history, \mathcal{B}_{-i} .

Lemma 13. *For the constructed \mathcal{B}_{-i} , $\mathbf{b}_1^{\text{OPT}}(\mathcal{B}_{-i}) = \mathbf{b}_1^*(\mathcal{B}_{-i}) = (1, 1)$ and $V_1^{\text{OPT}}(\mathcal{B}_{-i}) = V_1^*(\mathcal{B}_{-i}) = N^{m-1}$.*

Proof. We use a standard divide-and-conquer strategy to find the optimal 1-uniform bidding policy. In particular, we study the following problem: for a *fixed* q , what is the optimal value of b ? As q can be exponential in m , we exploit the structure of the bid history to compute the objective in an efficient manner.

Suppose $q = 1$. The maximum number of units (value) that can be obtained by bidding $(b, 1)$ is trivially N^{m-1} . Setting $b = 1$ achieves this upper bound. So, $(1, 1)$ obtains a total value N^{m-1} .

Suppose $q \geq 2$. Furthermore, assume $N^{m-j} < q \leq N^{m-j+1}$, for some $2 \leq j \leq m$.

- We first show that $b > w_{N^{m-j+1}} + \epsilon$ is not a feasible bid value for chosen values of q . Suppose $b > w_{N^{m-j+1}} + \epsilon$. Consider any round t in the partition j . Observe that bidding (b, q) obtains $q \geq N^{m-j} + 1$ units in this round as $b > w_{N^{m-j+1}} + \epsilon$, which is the least winning competing bid. Then, the clearing price is at least $w_{N^{m-j+1}} + \epsilon$ (which is the least winning competing bid for that round) but it is allocated exactly $N^{m-j} + 1$ units. Hence, the RoI constraint is violated.
- Next, we show that if $b < w_{N^{m-j+1}} + \epsilon$, the bidder is not allocated any units in several rounds. Specifically, if $b < w_{N^{m-j+1}} + \epsilon$, then (b, q) is not allocated any units in the partitions indexed by $j, j+1, \dots, m$ as b is strictly less than least winning competing bid.

So, if $N^{m-j} < q \leq N^{m-j+1}$, the optimal bidding policy (b, q) gets no units in

$$\sum_{\ell=j}^m N^{\ell-1} - N^{\ell-2} = N^{m-1} - N^{j-2} \text{ auctions.} \quad (41)$$

In the remaining N^{j-2} auctions it can win at most q units. So, the maximum value obtained by the optimal bidding policy (b, q) for any given $q \geq 2$ is

$$N^{j-2}(1 + (q-1)v) \leq N^{j-2}(1 + (N^{m-j+1} - 1)v) = N^{j-2} + (N^{m-1} - N^{j-2})v < N^{m-1},$$

where the last inequality holds as $v < 1$. Hence, $\mathbf{b}_1^{\text{OPT}}(\mathcal{B}_{-i}) = (1, 1)$. Moreover, observe that $\mathbf{b}_1^{\text{OPT}}(\mathcal{B}_{-i}) \in \mathcal{P}_1^*$, so $\mathbf{b}_1^{\text{OPT}}(\mathcal{B}_{-i}) = \mathbf{b}_1^*(\mathcal{B}_{-i})$ and $V_1^{\text{OPT}}(\mathcal{B}_{-i}) = V_1^*(\mathcal{B}_{-i}) = N^{m-1}$. \square

Substituting values from Lemma 12, and Lemma 13

$$\begin{aligned} \text{RATIO-OPT}_m &\geq \text{RATIO-OPT}_m(\mathcal{B}_{-i}) = \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_1^{\text{OPT}}(\mathcal{B}_{-i})} \\ &= \frac{N^{m-1} + (m-1)(N^{m-1} - N^{m-2})v}{N^{m-1}} \\ &= \frac{mN^{m-1} - (m-1)N^{m-2} - 2\epsilon'(m-1)(N^{m-1} - N^{m-2})}{N^{m-1}} \\ &= m - (m-1)\left(\frac{1}{N} + 2\epsilon'\left(1 - \frac{1}{N}\right)\right) \\ &= m - \delta, \end{aligned}$$

where the last step follows by substituting the value of ϵ' and thus concludes the proof. Similar analysis for RATIO-UNI_m also yields $\text{RATIO-UNI}_m \geq m - \delta$.

B.4 Tight Lower Bound for Theorem 6.2

In this section, we present a bid history \mathcal{B}_{-i} and valuation vector, \mathbf{v} such that for any $\delta \in (0, 1/2]$, $\frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_1^*(\mathcal{B}_{-i})} > 2m - \delta$. Recall that for showing tightness of the bound for Theorem 5.2 for any general m , we computed a lower bound to POUF_m of the form $\frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{mV_1^*(\mathcal{B}_{-i})}$. We consider a valuation curve and a bid history that has a structure identical to the one presented in Appendix B.2 but with the following modified parameters,

- Fix $m \geq 2$ and any integer $N \geq 2 \lceil \frac{2m}{\delta} \rceil$.
- Let $\bar{M} = N^{2m-1}$. Consider $T = N^{2m-1}$ rounds and $K = N^{2m-1} + 1$ units in each auction.
- Let $\epsilon' = \frac{\delta/(2m-1)-1/N}{2(1-1/N)} < \frac{\delta}{2} \leq \frac{1}{4}$. Set ϵ such that $\epsilon' = \epsilon N^{2m-1}(N^{2m-1} + 1)$.

Consider a valuation vector $\mathbf{v} = [1, v, \dots, v]$ such that $v = 1 - 2\epsilon'$, and target RoI $\gamma = 0$. Substituting the values from Lemma 10 and Lemma 11,

$$\begin{aligned} \max_{\mathcal{B}_{-i}} \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_1^*(\mathcal{B}_{-i})} &\geq \frac{V_m^{\text{OPT}}(\mathcal{B}_{-i})}{V_1^*(\mathcal{B}_{-i})} = \frac{N^{2m-1} + (2m-1)(N^{2m-1} - N^{2m-2})v}{N^{2m-1}} \\ &= \frac{2mN^{2m-1} - (2m-1)N^{2m-2} - 2\epsilon'(2m-1)(N^{2m-1} - N^{2m-2})}{N^{2m-1}} \\ &= 2m - (2m-1) \left(\frac{1}{N} + 2\epsilon' \left(1 - \frac{1}{N} \right) \right) \\ &= 2m - \delta, \end{aligned}$$

where the last step follows by substituting the value of ϵ' and thus concludes the proof.

C Simulation Details

C.1 Reconstructing Individual Bid Data

We obtained the publicly available auction data for $T_{\max} = 443$ EU ETS emission permit auctions held in 2022 and 2023 (EEX, 2023). For each auction indexed by $t \in [T_{\max}]$, we have the following relevant information: the minimum bid (b_{\min}^t), the maximum bid (b_{\max}^t), the average of the bids (b_{avg}^t), the median of the bids (b_{med}^t), and the number of bid-quantity pairs submitted (n_{sub}^t). We normalized the bids to be in $[0, 1]$. For all rounds t , $b_{\text{avg}}^t \approx b_{\text{med}}^t$ (linear regression yields coefficient 1.01 and intercept -0.008).

Upon further investigation, we observed that, except a few, a significant number of auctions had either $b_{\min}^t \approx b_{\text{avg}}^t \ll b_{\max}^t$ (Type I) or $b_{\min}^t \ll b_{\text{avg}}^t \approx b_{\max}^t$ (Type II). As $b_{\text{avg}}^t \approx b_{\text{med}}^t, \forall t$, we deduce that for Type I, most of the bids are concentrated in the interval $[b_{\min}^t, 2b_{\text{avg}}^t - b_{\min}^t]$ whereas for Type II, most of the bids are in the interval $[2b_{\text{avg}}^t - b_{\max}^t, b_{\max}^t]$. We posit that for Type I (resp. Type II) auctions, $f \in (0, 1)$ fraction of the bids (n_{sub}^t) are in $[b_{\min}^t, 2b_{\text{avg}}^t - b_{\min}^t]$ (resp. $[2b_{\text{avg}}^t - b_{\max}^t, b_{\max}^t]$) and the $1 - f$ fraction of bids are in $[2b_{\text{avg}}^t - b_{\min}^t, b_{\max}^t]$ (resp. $[b_{\min}^t, 2b_{\text{avg}}^t - b_{\max}^t]$). If for Type I (resp. Type II) auctions, $2b_{\text{avg}}^t > b_{\min}^t + b_{\max}^t$ (resp. $2b_{\text{avg}}^t < b_{\min}^t + b_{\max}^t$), we assume that all the bids are uniformly present in the interval $[b_{\min}^t, b_{\max}^t]$. With these assumptions, we generate individual bid data for each auction by sampling uniformly from these intervals.

After generating the individual bid data, we compute the metrics for the reconstructed bids (say \hat{b}_{avg}^t) for each auction and reject those with a relative error of at least δ (tolerance). For our

simulations, we set $\delta = 0.05$. We perform a grid search for f to maximize the number of auctions where the metrics of the reconstructed data are within δ relative error of the actual metrics, and obtain that $f = 0.97$. Following this pre-processing, we have reconstructed individual bid data for $T = 341$ auctions. This reconstructed data is used in our simulations in Section 8.

C.2 Formulating the Integer Linear Program

Characterizing (VM-UNI_m). By Corollary 7.1, we have that for any $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$,

$$\max_{\mathbf{b} \in \mathcal{P}_m^*} \sum_{t=1}^T V_i(\mathbf{b}, \beta_{-i}^t) = \max_{S \subseteq [\bar{M}]: |S| \leq m} \sum_{t=1}^T \max_{\ell \in S} V_i \left(\left(\frac{w_\ell}{1+\gamma}, \ell \right), \beta_{-i}^t \right).$$

Let $v_{j,t} = V_i \left(\left(\frac{w_j}{1+\gamma}, j \right), \beta_{-i}^t \right)$, i.e., the value obtained by bidding $\left(\frac{w_j}{1+\gamma}, j \right)$ in round t for any $j \in [\bar{M}]$ and $t \in [T]$. Then, (VM-UNI_m) can be equivalently formulated as:

$$\begin{aligned} \max \quad & \sum_{t=1}^T \sum_{j=1}^{\bar{M}} v_{j,t} \cdot y_{j,t} && \text{(VM-UNI}_m\text{-ILP)} \\ \text{such that} \quad & \sum_{j=1}^{\bar{M}} y_{j,t} = 1, && \forall t \in [T] \\ & y_{j,t} \leq x_j, && \forall t \in [T], j \in [\bar{M}] \\ & \sum_{j=1}^{\bar{M}} x_j \leq m, && \\ & x_j, y_{j,t} \in \{0, 1\}. && \forall t \in [T], j \in [\bar{M}] \end{aligned}$$

We note that the formulation in Eq. (VM-UNI_m-ILP) is a special case of the m -facility location problem (Cornuéjols et al., 1983) and is identical to the float maximization problem in bank accounts (Cornuéjols et al., 1977; Williamson and Shmoys, 2011, pp. 47). Both the problems are not known to have any polynomial time algorithm that results in an exact optimal solution.

An Uniform Upper Bound for $V_m^{\text{OPT}}(\mathcal{B}_{-i})$. For any bid history, $\mathcal{B}_{-i} = [\beta_{-i}^t]_{t \in [T]}$, suppose $\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i})$ is allocated r_t units in any round t . Then, by Lemma 9 (1), we know that $\left(\frac{w_{r_t}}{1+\gamma}, r_t \right)$ also gets r_t units in round t . So,

$$\begin{aligned} V_i(\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i}); \beta_{-i}^t) &= V_i \left(\left(\frac{w_{r_t}}{1+\gamma}, r_t \right); \beta_{-i}^t \right) \leq \max_{\mathbf{b} \in \mathcal{P}_1^*} V_i(\mathbf{b}; \beta_{-i}^t) \\ \implies V_m^{\text{OPT}}(\mathcal{B}_{-i}) &= \sum_{t=1}^T V_i(\mathbf{b}_m^{\text{OPT}}(\mathcal{B}_{-i}); \beta_{-i}^t) \leq \sum_{t=1}^T \max_{\mathbf{b} \in \mathcal{P}_1^*} V_i(\mathbf{b}; \beta_{-i}^t) = \sum_{t=1}^T \max_{j \in [\bar{M}]} v_{j,t}, \end{aligned}$$

which is the optimal objective value of (VM-UNI_m-ILP), when solved without the cardinality constraints, i.e., $\sum_{j=1}^{\bar{M}} x_j \leq m$.

Contact.

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Center for Energy and
Environmental Policy Research

**MIT Center for Energy and
Environmental Policy Research**
Massachusetts Institute of Technology
77 Massachusetts Avenue, E19-411
Cambridge, MA 02139-4307
USA

ceepr.mit.edu



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